ECEN 460, Spring 2026
Power System Operation and Control

Class 15: Power Flow Approximations

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Announcements

- Homework #6: book problems 6.9, 6.12, 6.18, 6.25, 6.38, due Thursday, Mar 5, 2026

Modeling Voltage Dependent Load

- So far we've assumed that the load is independent of the bus voltage (i.e., constant power). However, the power flow can be easily extended to include voltage dependence with both the real and reactive load.
- This is done by making P_{Di} and Q_{Di} a function of $|V_i|$

$$\sum_{k=1}^{n} |V_i||V_k| \left(G_{ik} \cos\theta_{ik} + B_{ik} \sin\theta_{ik}\right) - P_{Gi} + P_{Di}(|V_i|) = 0$$

$$\sum_{k=1}^{n} |V_i||V_k| \left(G_{ik} \sin\theta_{ik} - B_{ik} \cos\theta_{ik}\right) - Q_{Gi} + Q_{Di}(|V_i|) = 0$$

Voltage Dependent Load Example

- In the two-bus example assume the load is constant impedance, so that

$$\begin{aligned} \mathrm{P}_2(\mathbf{x}) \; &= \; |V_2|(10\sin\theta_2) + 2.0|V_2|^2 \; = \; 0 \\ Q_2(\mathbf{x}) \; &= \; |V_2|(-10\cos\theta_2) + |V_2|^2(10) + 1.0|V_2|^2 = 0 \end{aligned}$$

- Now calculate the power flow Jacobian

$$J(\mathbf{x}) \; = \begin{bmatrix} 10|V_2|\cos\theta_2 & 10\sin\theta_2 + 4.0|V_2| \\ 10|V_2|\sin\theta_2 & -10\cos\theta_2 + 20|V_2| + 2.0|V_2| \end{bmatrix}$$

Voltage Dependent Load, cont'd

- Again set $v = 0$, guess $\mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Calculate

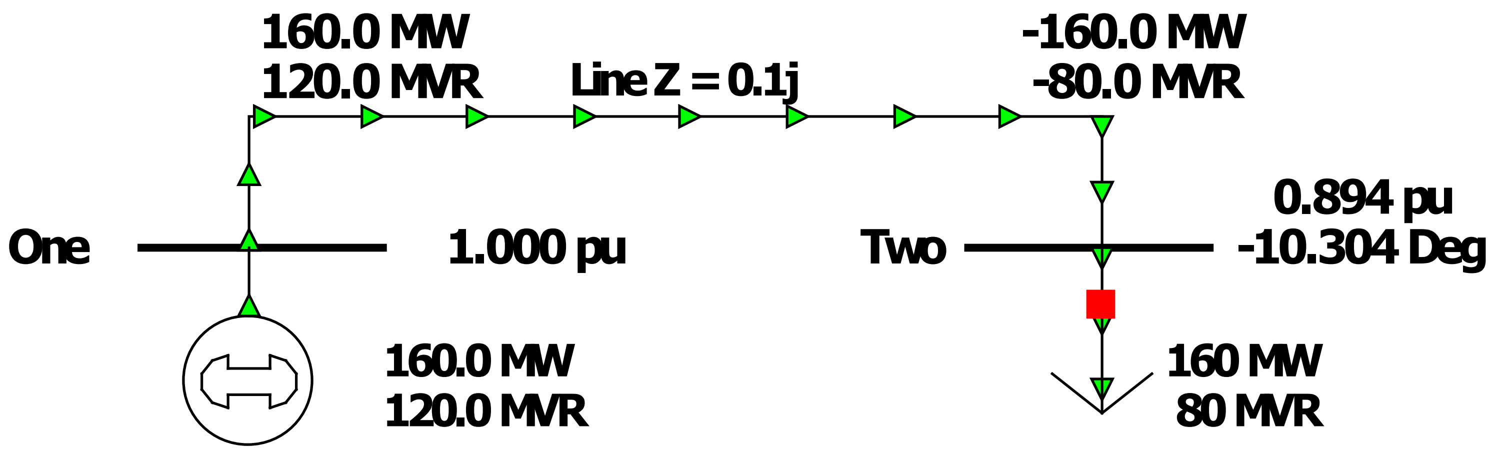
$$f\left(\mathbf{x}^{(0)}\right) = \begin{bmatrix} |V_2|(10\,\sin\theta_2) + 2.0|V_2|^2 \\ |V_2|(-10\,\cos\theta_2) + |V_2|^2(10) + 1.0|V_2|^2 \end{bmatrix} = \begin{bmatrix} 2.0 \\ 1.0 \end{bmatrix}$$

$$\mathbf{J}(\mathbf{x}^{(0)}) = \begin{bmatrix} 10 & 4 \\ 0 & 12 \end{bmatrix}$$

Solve $\mathbf{x}^{(1)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 10 & 4 \\ 0 & 12 \end{bmatrix}^{-1} \begin{bmatrix} 2.0 \\ 1.0 \end{bmatrix} = \begin{bmatrix} -0.1667 \\ 0.9167 \end{bmatrix}$

Voltage Dependent Load, cont'd

- With constant impedance load the MW/Mvar load at bus 2 varies with the square of the bus 2 voltage magnitude. This if the voltage level is less than 1.0, the load is lower than 200/100 MW/Mvar



Speeding up the Power Flow

- The most computationally complex part of power flow is building and inverting (or factorizing) the Jacobian, so most of the approximations are designed to speed it up by reducing that work

Power Flow Approximations

- Methods used to approximate the power flow and potentially speed it up:
  - **Dishonest Newton-Raphson**: just keep the same Jacobian from the first iteration to use in all iterations. (Or only update occasionally.)
  - **Decoupled Power Flow**: Assume the off-diagonal Jacobian matrices are zero (they are usually quite small)
  - **Fast Decoupled Power Flow**: Same as decoupled but only build and invert the Jacobian once (like in the Dishonest Newton-Raphson method)
  - **Linearized or "DC" power flow**: ignore reactive power completely and assume all bus voltages are 1.0 per-unit.

Dishonest Newton-Raphson

- Since most of the time in the Newton-Raphson iteration is spend calculating the inverse of the Jacobian, one way to speed up the iterations is to only calculate/inverse the Jacobian occasionally
  - known as the "Dishonest" Newton-Raphson
  - an extreme example is to only calculate the Jacobian for the first iteration

Honest: $\mathbf{x}^{(v+1)} = \mathbf{x}^{(v)} - \mathbf{J}(\mathbf{x}^{(v)})^{-1}\mathbf{f}(\mathbf{x}^{(v)})$

Dishonest: $\mathbf{x}^{(v+1)} = \mathbf{x}^{(v)} - \mathbf{J}(\mathbf{x}^{(0)})^{-1}\mathbf{f}(\mathbf{x}^{(v)})$

Both require $\left\|\mathbf{f}(\mathbf{x}^{(v)})\right\| < \varepsilon$ for a solution

Dishonest Newton-Raphson Example

- Use the Dishonest Newton−Raphson to solve

$$f(x) = x^2 - 2 = 0$$

$$\Delta x^{(v)} = -\left[\frac{df(x^{(0)})}{dx}\right]^{-1} f(x^{(v)})$$

$$\Delta x^{(v)} = -\left[\frac{1}{2x^{(0)}}\right]((x^{(v)})^2 - 2)$$

$$x^{(v+1)} = x^{(v)} - \left[\frac{1}{2x^{(0)}}\right]((x^{(v)})^2 - 2)$$

Dishonest N-R Example, cont'd

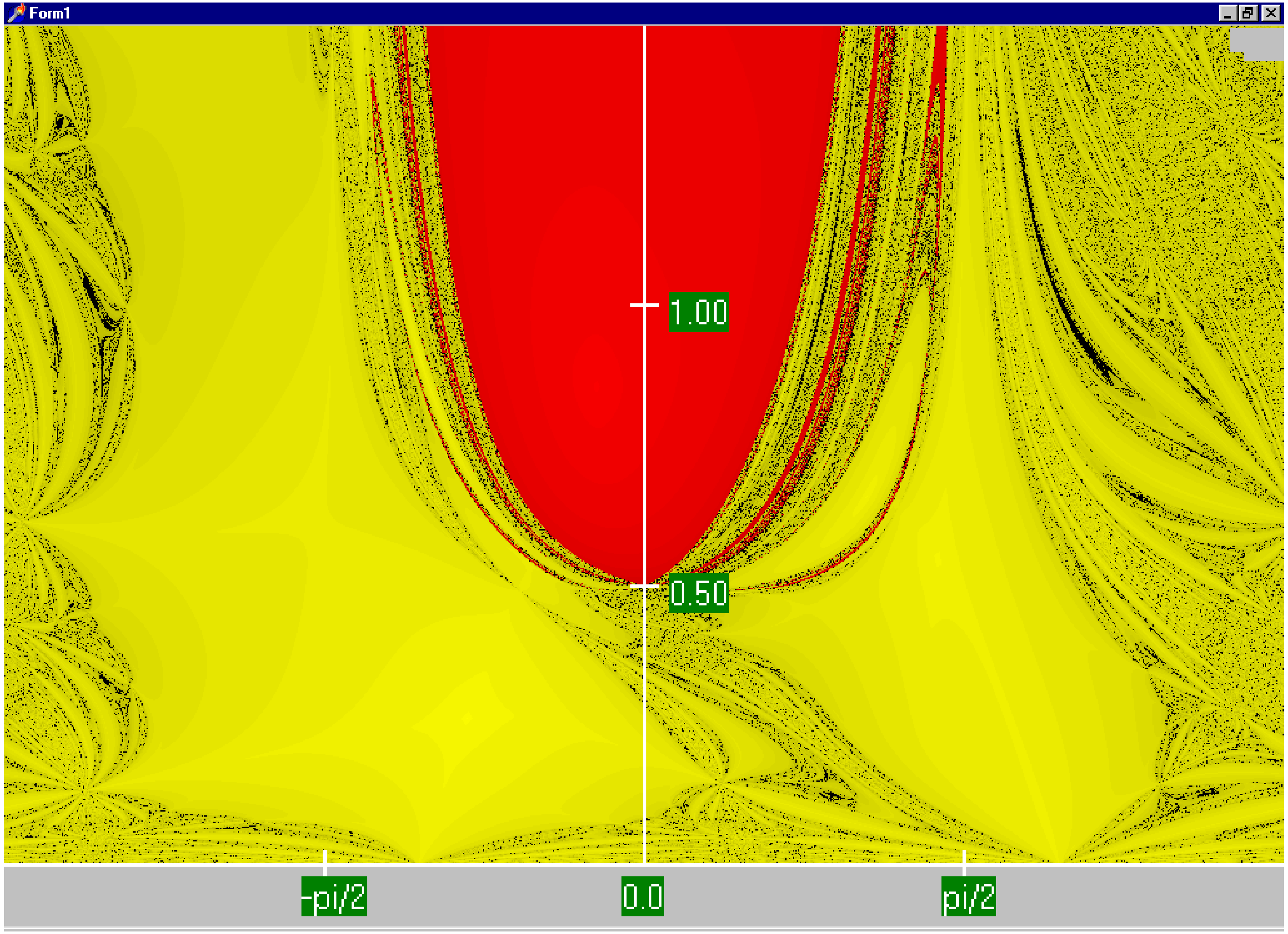
- $x^{(v+1)} = x^{(v)} - \left[\frac{1}{2x^{(0)}}\right]((x^{(v)})^2 - 2)$

Guess $\mathrm{x}^{(0)} = 1.$ Iteratively solving we get

ν	$x^{(\nu)}$ Honest	$x^{(\nu)}$ Dishonest
0	1	1
1	1.5	1.5
2	1.41667	1.375
3	1.41422	1.429
4	1.41422	1.408

We pay a price in increased iterations, but with decreased computation per iteration

Regular Newton-Raphson Region of Convergence

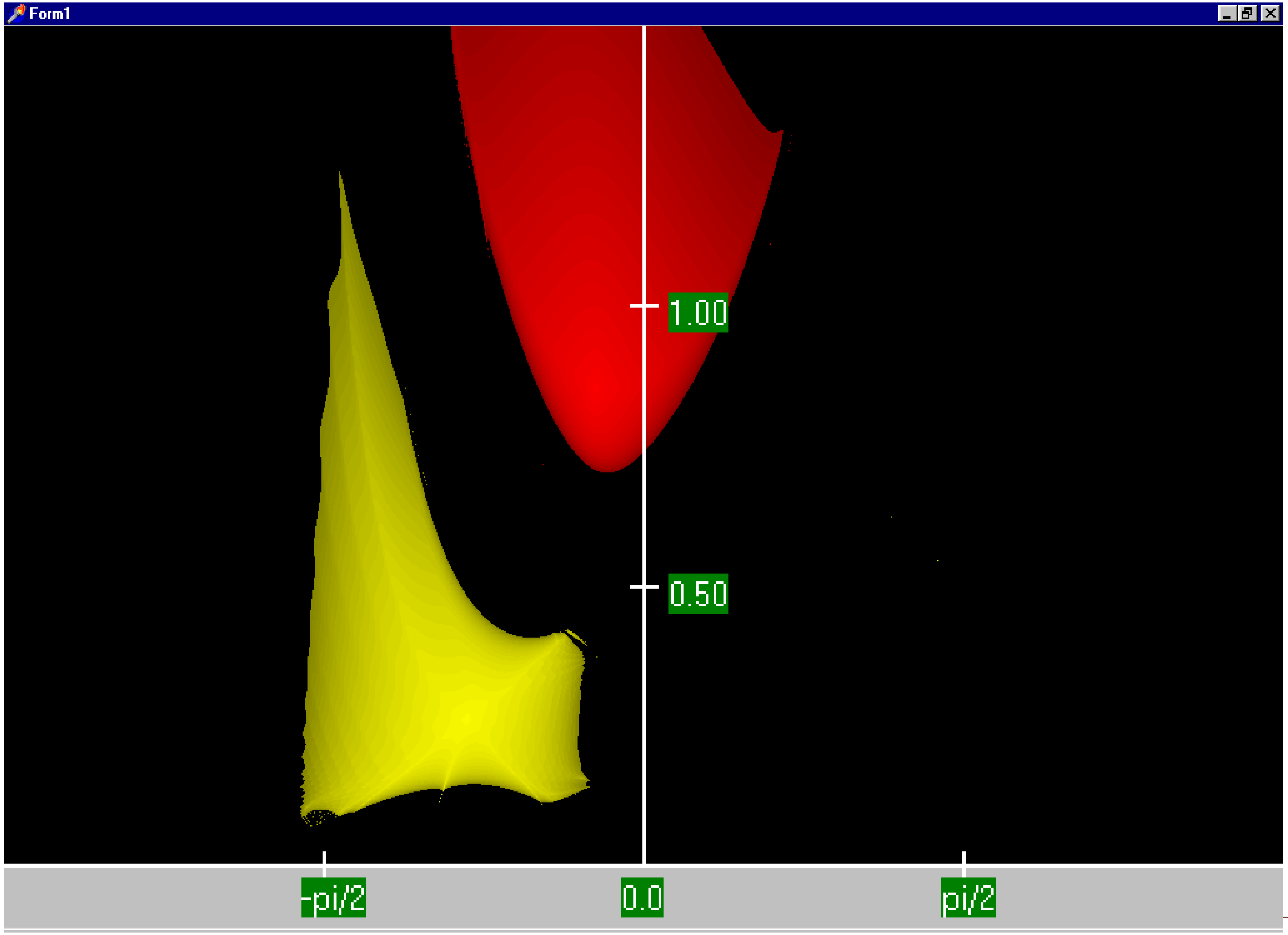


For a two-bus system, this plot is graphing the initial guess: horizontal axis is theta and vertical axis is V.

Red region converges to the high voltage solution, while the yellow region converges to the low voltage solution

Maximum of 15 iterations

Dishonest Newton-Raphson Region of Convergence



For a two-bus system, this plot is graphing the initial guess: horizontal axis is theta and vertical axis is V.

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Maximum of 15 iterations

Decoupled Power Flow

- The completely Dishonest Newton-Raphson is not used for power flow analysis. However several approximations of the Jacobian matrix are used.
- One common method is the decoupled power flow. In this approach approximations are used to decouple the real and reactive power equations.

Decoupled Power Flow Formulation

- General form of the power flow problem

$$-\begin{bmatrix} \frac{\partial \mathbf{P}}{\partial \boldsymbol{\theta}}^{(v)} & \frac{\partial \mathbf{P}}{\partial |\mathbf{V}|}^{(v)} \\ \frac{\partial \mathbf{Q}}{\partial \boldsymbol{\theta}}^{(v)} & \frac{\partial \mathbf{Q}}{\partial |\mathbf{V}|}^{(v)} \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{\theta}^{(v)} \\ \Delta|\mathbf{V}|^{(v)} \end{bmatrix} = \begin{bmatrix} \Delta\mathbf{P}(\mathbf{x}^{(v)}) \\ \Delta\mathbf{Q}(\mathbf{x}^{(v)}) \end{bmatrix} = \mathbf{f}(\mathbf{x}^{(v)})$$

- where

$$\Delta\mathbf{P}(\mathbf{x}^{(v)}) = \begin{bmatrix} P_2(\mathbf{x}^{(v)}) + P_{D2} - P_{G2} \\ \vdots \\ P_n(\mathbf{x}^{(v)}) + P_{Dn} - P_{Gn} \end{bmatrix}$$

Decoupling Approximation

- Usually the off−diagonal matrices, $\frac{\partial \mathbf{P}}{\partial |\mathbf{V}|}^{(v)}$ and $\frac{\partial \mathbf{Q}}{\partial \boldsymbol{\theta}}^{(v)}$ are small. Therefore we approximate them as zero:

$$-\begin{bmatrix} \frac{\partial \mathbf{P}}{\partial \boldsymbol{\theta}}^{(v)} & \mathbf{0} \\ \mathbf{0} & \frac{\partial \mathbf{Q}}{\partial |\mathbf{V}|}^{(v)} \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{\theta}^{(v)} \\ \Delta|\mathbf{V}|^{(v)} \end{bmatrix} = \begin{bmatrix} \Delta\mathbf{P}(\mathbf{x}^{(v)}) \\ \Delta\mathbf{Q}(\mathbf{x}^{(v)}) \end{bmatrix} = \mathbf{f}(\mathbf{x}^{(v)})$$

- Then the problem can be decoupled

$$\Delta\boldsymbol{\theta}^{(v)} = -\left[\frac{\partial \mathbf{P}}{\partial \boldsymbol{\theta}}^{(v)}\right]^{-1} \Delta\mathbf{P}\left(\mathbf{x}^{(v)}\right)$$

$$\Delta|\mathbf{V}|^{(v)} = -\left[\frac{\partial \mathbf{Q}}{\partial |\mathbf{V}|}^{(v)}\right]^{-1} \Delta\mathbf{Q}(\mathbf{x}^{(v)})$$

Off-diagonal Jacobian Terms

- Justification for Jacobian approximations:

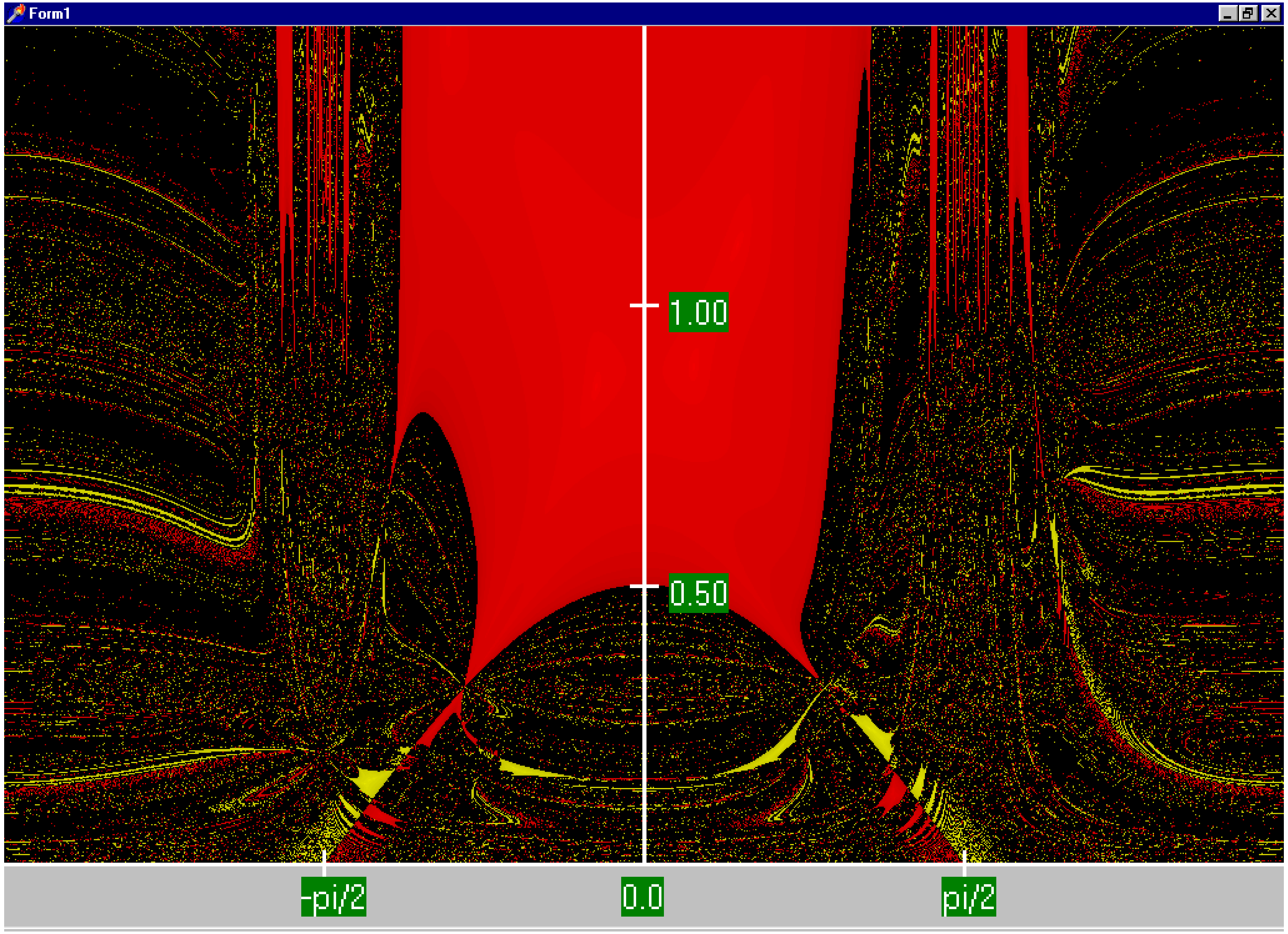
1. Usually r $\ll$ x, therefore $|G_{ij}| \ll |B_{ij}|$
2. Usually θ_{ij} is small so $\sin\theta_{ij} \approx 0$

- Therefore

$$\frac{\partial \mathbf{P}_i}{\partial |\mathbf{V}_j|} = |V_i|\left(G_{ij}\cos\theta_{ij} + B_{ij}\sin\theta_{ij}\right) \approx 0$$

$$\frac{\partial \mathbf{Q}_i}{\partial \boldsymbol{\theta}_j} = -|V_i||V_j|\left(G_{ij}\cos\theta_{ij} + B_{ij}\sin\theta_{ij}\right) \approx 0$$

Decoupled Power Flow Region of Convergence



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Red region converges to the high voltage solution, while the yellow region converges to the low voltage solution

Maximum of 15 iterations

Fast Decoupled Power Flow

- By continuing with our Jacobian approximations we can actually obtain a reasonable approximation that is independent of the voltage magnitudes/angles.
- The Jacobian need only be built/inverted once.
- This approach is known as the fast decoupled power flow (FDPF)
- FDPF uses the same mismatch equations as standard power flow so it should have same solution
- The FDPF is sometimes used, particularly when we only need an approximate solution such as in contingency analysis

"DC" Power Flow

- The "DC" power flow makes the most severe approximations:
  - completely ignore reactive power, assume all the voltages are always 1.0 per unit, ignore line conductance
- This makes the power flow a linear set of equations, which can be solved directly

$$\boldsymbol{\theta} = \mathbf{B}^{-1}\ \mathbf{P}$$

- The advantage is it is fast, and it has a guaranteed solution. The disadvantage is the degree of approximation. However, it is used sometimes.
- Note that even though it is called "DC" we are still solving an AC system. This is different from including HVDC lines in the power flow.

DC Power Flow Example

EXAMPLE 6.17

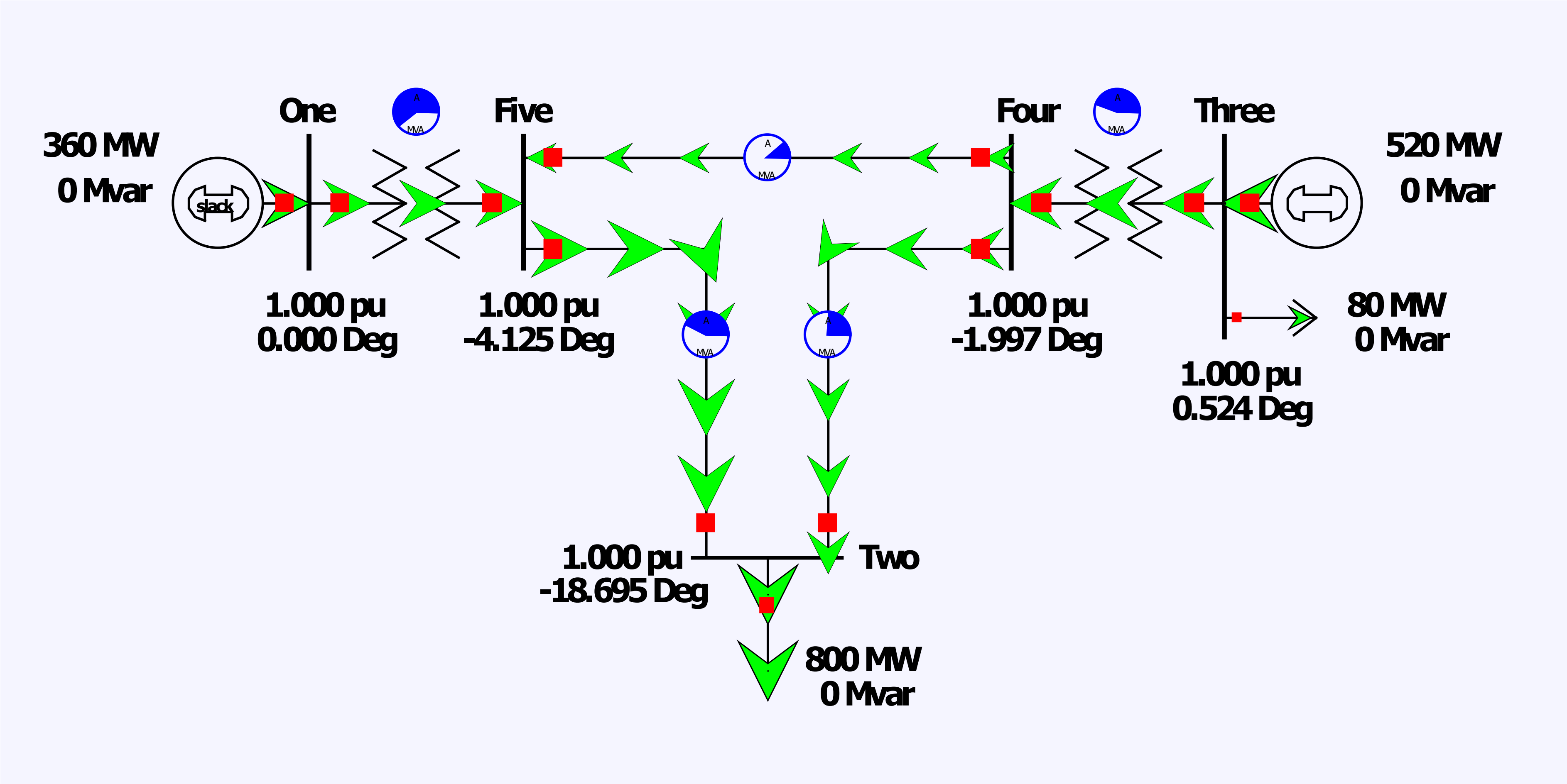
Determine the dc power flow solution for the five bus from Example 6.9.

SOLUTION With bus 1 as the system slack, the **B** matrix and **P** vector for this system are

$$\mathbf{B} = \begin{bmatrix} -30 & 0 & 10 & 20 \\ 0 & -100 & 100 & 0 \\ 10 & 100 & -150 & 40 \\ 20 & 0 & 40 & -110 \end{bmatrix} \quad \mathbf{P} = \begin{bmatrix} -8.0 \\ 4.4 \\ 0 \\ 0 \end{bmatrix}$$

$$\delta = -\mathbf{B}^{-1}\mathbf{P} = \begin{bmatrix} -0.3263 \\ 0.0091 \\ -0.0349 \\ -0.0720 \end{bmatrix} \text{radians} = \begin{bmatrix} -18.70 \\ 0.5214 \\ -2.000 \\ -4.125 \end{bmatrix} \text{degrees}$$

DC Power Flow 5 Bus Example



Notice with the dc power flow all of the voltage magnitudes are 1 per unit.

Example Algorithm: Left-looking LU Factorization

- Computing L and U one column at a time. In below matrix, middle column and row are row k that is currently being computed:

$$\begin{bmatrix} L_{11} & & \\ \ell_{21} & 1 & \\ L_{31} & \ell_{32} & L_{33} \end{bmatrix} \begin{bmatrix} U_{11} & u_{12} & U_{13} \\ & u_{22} & u_{23} \\ & & U_{33} \end{bmatrix} = \begin{bmatrix} A_{11} & a_{12} & A_{13} \\ a_{21} & a_{22} & a_{23} \\ A_{31} & a_{32} & A_{33} \end{bmatrix}$$

- Assume all of A is known, and we have already computed left columns of LU
- Then we can solve for $L_{11}u_{12} = a_{12}$, $\ell_{21}u_{12} + u_{22} = a_{22}$, and $L_{31}u_{21} + \ell_{32}u_{22} = a_{32}$
- Or equivalently, solve the sparse triangular system:

$$\begin{bmatrix} L_{11} & & \\ \ell_{21} & 1 & \\ L_{31} & 0 & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}$$

- Then set $u_{12} = x_1$, $u_{22} = x_2$, and $\ell_{32} = x_3/u_{22}$

A DC Power Flow Example

$$\boldsymbol{\theta} = -\mathbf{B}^{-1}\ \mathbf{P}$$

- Here were the impedances we found, in per-unit on the system base:
  - $Z_{xf12} = 0.0186 + j0.187$
  - $Z_{line23} = 0.054 + j0.1$
  - $Z_{xf34} = 0.027 + j0.216$
  - Note: we are modeling the generator as the slack bus (no impedance) and the load as constant power 20 MW and 10 Mvar
- Solve the system with the DC power flow

A DC Power Flow Example, cont.

1. Make the y-bus matrix

$$Y_{bus} = \begin{bmatrix} 0.527 - j5.30 & -0.527 + j5.30 & 0 & 0 \\ -0.527 + j5.30 & 4.71 - j13.04 & -4.18 + j7.74 & 0 \\ 0 & -4.18 + j7.74 & 4.75 - j12.30 & -0.570 + j4.56 \\ 0 & 0 & -0.570 + j4.56 & 0.570 - j4.56 \end{bmatrix}$$

2. Find the B matrix as the imaginary part of Y without the slack bus row/column

$$B = \begin{bmatrix} -13.04 & 7.74 & 0 \\ 7.74 & -12.30 & 4.56 \\ 0 & 4.56 & -4.56 \end{bmatrix}$$

3. Find the P-vector of power injections and solve $\boldsymbol{\theta} = -\mathbf{B}^{-1}\ \mathbf{P}$

$$\theta = -\begin{bmatrix} -13.04 & 7.74 & 0 \\ 7.74 & -12.30 & 4.56 \\ 0 & 4.56 & -4.56 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.0377 \\ 0.0636 \\ 0.1074 \end{bmatrix}$$

Solution:

$V_1 = 1\angle 0°$

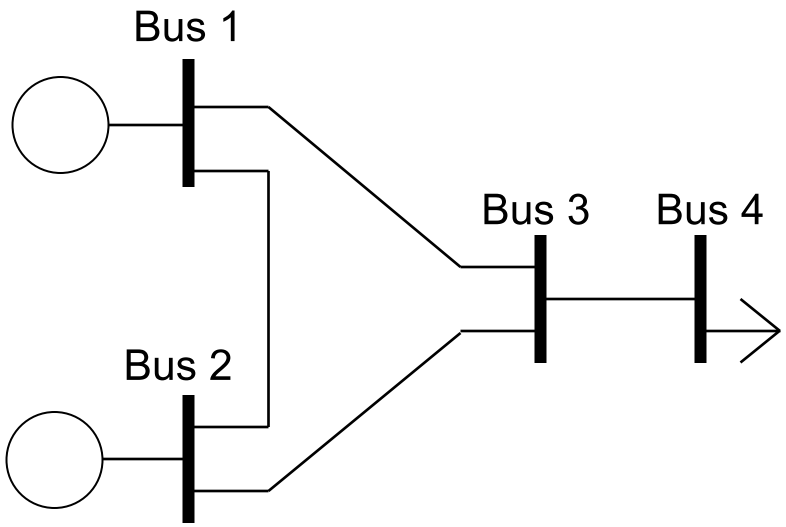
$V_2 = 1\angle 2.16°$

$V_3 = 1\angle 3.64°$

$V_4 = 1\angle 6.16°$

Another Power Flow Example

- For this four-bus power flow problem
  - Make the Y-bus matrix
  - Identify the bus type and known/unknown variables for each bus
  - Write the P and Q equations for bus 4
  - Write the last row (Q_4) of the Jacobian matrix
- Solve the dc power flow for this problem
  - What is the approximate power flow on the line 1-2?



Bus 1: Generator with voltage setpoint of 1.00 pu

Bus 2: Generator with voltage setpoint of 1.02 pu and power setpoint of 75 MW

Bus 4: Load with 180 MW and 55 Mvar

Line 1-3: Z = j0.25

Line 1-2: Z = j0.05

Line 2-3: Z = j0.2

Line 3-4: Z = 0.04+j0.08

Another Power Flow Example, Cont.

$$\boldsymbol{\theta} = \mathbf{B}^{-1}\ \mathbf{P}$$

- Y-bus matrix

$$\begin{bmatrix} -j24 & j20 & j4 & 0 \\ j20 & -j25 & j5 & 0 \\ j4 & j5 & 5-j19 & -5+j10 \\ 0 & 0 & -5+j10 & 5-j10 \end{bmatrix}$$

$$\mathrm{P_i} = \sum_{k=1}^{n} |V_i||V_k|\,(g_{ik}\cos\theta_{ik} + b_{ik}\sin\theta_{ik}) = P_{Gi} - P_{Di}$$

$$\mathrm{Q_i} = \sum_{k=1}^{n} |V_i||V_k|\,(g_{ik}\sin\theta_{ik} - b_{ik}\cos\theta_{ik}) = Q_{Gi} - Q_{Di}$$

- Bus 1 is slack (unknown P/Q)
- Bus 2 is PV (unknown θ/Q)
- Buses 3&4 are PQ (unkn. θ/V)
- $P_4 = -1.8 = V_4V_3(-5\cos\theta_{43} + 10\sin\theta_{43}) + 5V_4^2$
- $Q_4 = -0.55 = V_4V_3(-5\sin\theta_{43} - 10\cos\theta_{43}) - 10V_4^2$
- In last row of Jacobian: $\frac{\partial Q_4}{\partial \theta_3} = V_4V_3(5\cos\theta_{43} - 10\sin\theta_{43}); \frac{\partial Q_4}{\partial \theta_4} = -\frac{\partial Q_4}{\partial \theta_3}$
- $\frac{\partial Q_4}{\partial V_3} = V_4(-5\sin\theta_{43} - 10\cos\theta_{43}); \frac{\partial Q_4}{\partial V_4} = V_3(-5\sin\theta_{43} - 10\cos\theta_{43}) - 20V_4$

Another Power Flow Example, dc part

$\boldsymbol{\theta} = \mathbf{B}^{-1}\ \mathbf{P}$

- B matrix – imaginary part of y-bus without slack row and column

$$\begin{bmatrix} -25 & 5 & 0 \\ 5 & -19 & 10 \\ 0 & 10 & -10 \end{bmatrix}$$

- P vector based on power leaving non-slack buses

$$P = \begin{bmatrix} -0.75 \\ 0 \\ 1.8 \end{bmatrix}$$

- Solve for theta (remember this is an approximation)

$$\theta = B^{-1}P = \begin{bmatrix} -0.01125 \\ -0.20625 \\ -0.38625 \end{bmatrix}$$

- Get line flow based on dc assumptions

$P_{12} = (-B_{Line})(\theta_1 - \theta_2) = 20(0 - -0.01125) = 0.225$ p.u. (or 22.5 MW)

Inverse of a Sparse Matrix

- The inverse of a sparse matrix is NOT in general a sparse matrix
- We never (or at least very, very, very seldom) explicitly invert a sparse matrix
  - Individual columns of the inverse of a sparse matrix can be obtained by solving $x = A^{-1}b$ with b set to all zeros except for a single nonzero in the position of the desired column
  - If a few desired elements of A^{-1} are desired (such as the diagonal values) they can usually be computed quite efficiently using sparse vector methods
- We can't invert a singular matrix (with sparse or not)

Brief Coverage of Sparse Matrices

- A problem that occurs in many is fields is the solution of linear systems

  Ax = b

  where

  A is an n by n matrix with elements a_{ij},

  and x and b are n-vectors with elements x_i and b_i respectively
- In power systems we are particularly interested in systems when n is relatively large and A is sparse
  - How large is large is changing
- A matrix is sparse if a large percentage of its elements have zero values
- Goal is to be aware of the computational issues (including complexity) associated with the solution of these systems

Brief Coverage of Sparse Matrices

- Sparse matrices arise in many areas, and can have domain specific structures
- Much of the early sparse matrix work was done in power!
- Great book on modern methods for solving sparse matrices is by Texas A&M professor Tim Davis