ECEN 460, Spring 2026
Power System Operation and Control

Class 12: The Power Flow Problem

Prof. Adam Birchfield
Dept. of Electrical and Computer Engineering
Texas A&M University
abirchfield@tamu.edu

Announcements

- Homework #5: book problems 6.30 and 6.31, due Thursday, Feb. 26, 2026
- Homework #6: book problems 6.9, 6.12, 6.18, 6.25, 6.38, due Thursday, Mar 5, 2026

Recall, Transmission Line and Transformer Models

- Transmission lines use the "pi" model

These devices reduce down to impedances

- Transformer non-ideal model is simplified with the per-unit system (and neglecting shunt elements)

Recall, Building the Y-Bus Matrix

- Here are the steps for building the Y-bus matrix
  - Start with an NxN matrix of all zeros (N is the number of buses)
  - For any shunt elements (connected to ground, such as most capacitors)
    - Add $\bar{Y}$ to the corresponding diagonal elements of the Y-bus
  - For branches between two buses,
    - Add $\bar{Y}$ to both diagonal elements of the Y-bus
    - Subtract $\bar{Y}$ from both off-diagonal elements of the Y-bus
- We add one network element at a time to the Y-bus matrix!

Remember, $\bar{Y} = 1/\bar{Z}$

$$\longrightarrow \begin{bmatrix} \bar{Y}_A + \bar{Y}_B & -\bar{Y}_A & -\bar{Y}_B & 0 \\ -\bar{Y}_A & \bar{Y}_A + \bar{Y}_C + \bar{Y}_D & -\bar{Y}_C & -\bar{Y}_D \\ -\bar{Y}_B & \bar{Y}_C & \bar{Y}_B + \bar{Y}_C & 0 \\ 0 & -\bar{Y}_D & 0 & \bar{Y}_D \end{bmatrix}$$

The Power Flow Problem

- What we know:
  - Generator real power
  - Generator voltage magnitude
  - Load real and reactive power
- What we don't know:
  - Bus voltage angles
  - Non-generator bus voltage magnitude
  - Generator reactive power
  - Current injections of generators or loads
- Note that the problem is formulated with *power* values, not current, so we cannot use the Y-bus equations directly
- Power flow problem is ***non-linear***

Example of a Non-Linear Problem

Solve for x: $x - \sqrt{x} - 1 = 0$

- Can solve *analytically* or *iteratively*
  - *Analytically*: manipulate the algebra to attempt to isolate *x* and get a closed-form solution (possible but difficult here, may have hidden assumptions, impossible in some problems)
  - *Iteratively*: make a guess at the answer and use a systematic updating technique to gradually improve the guess
    - May ***converge*** or ***diverge***
    - Convergence depends on how good the initial guess is
    - Two methods we will discuss: **Gauss** and **Newton-Raphson**

The analytical solution of this problem is

$1.5 + 0.5\sqrt{5}$

Gauss Method

Method:

1. Write equation in implicit form x=h(x)
2. Make an initial guess of x, $x^{(0)}$
3. For each iteration ν, $x^{(\nu+1)} = h\left(x^{(\nu)}\right)$
4. When a fixed point is found such that $x^{(\nu+1)} = x^{(\nu)}$, solution to original equation has been found

Gauss Method 2

Method:

1. Write equation in implicit form x=h(x)
2. Make an initial guess of x, $x^{(0)}$
3. For each iteration ν, $x^{(\nu+1)} = h\left(x^{(\nu)}\right)$
4. When a fixed point is found such that $x^{(\nu+1)} = x^{(\nu)}$, solution to original equation has been found

Solve $x - \sqrt{x} - 1 = 0$

$$x^{(\nu)} = 1 + \sqrt{x^{(\nu-1)}}$$

Guess $x^{(0)} = 1$

ν	$x^{(\nu)}$
0	1
1	2
2	2.41421
3	2.55377
4	2.59805
5	2.61185

ν	$x^{(\nu)}$
6	2.61612
7	2.61744
8	2.61785
9	2.61798
10	2.61802
11	2.61803

Newton-Raphson Method

Method:

1. Write equation as f($\hat{x}$)=0
2. Make an initial guess of $\hat{x}$, $x^{(0)}$
3. For iteration ν, define $\Delta x = \hat{x} - x^{(\nu)}$
4. Write Taylor series expansion:

$$f(\hat{x}) = f\left(x^{(\nu)}\right) + \frac{df\left(x^{(\nu)}\right)}{dx}\Delta x^{(\nu)} + \text{higher order terms}$$

5. Approximate with only first two terms:

$$0 = f\left(x^{(\nu)}\right) + \frac{df\left(x^{(\nu)}\right)}{dx}\Delta x^{(\nu)}$$

6. Solve for $\Delta x^{(\nu)} = -\left[\frac{df\left(x^{(\nu)}\right)}{dx}\right]^{-1} f\left(x^{(\nu)}\right)$
7. Solve for new estimate of $\hat{x}$

$$x^{(\nu+1)} = x^{(\nu)} + \Delta x^{(\nu)}$$

Newton-Raphson Method, example

Method:

1. Write equation as f($\hat{x}$)=0
2. Make an initial guess of $\hat{x}$, $x^{(0)}$
3. For iteration ν, define $\Delta x = \hat{x} - x^{(\nu)}$
4. Write Taylor series expansion:

$$f(\hat{x}) = f\left(x^{(\nu)}\right) + \frac{df\left(x^{(\nu)}\right)}{dx}\Delta x^{(\nu)} + \text{higher order terms}$$

5. Approximate with only first two terms:

$$0 = f\left(x^{(\nu)}\right) + \frac{df\left(x^{(\nu)}\right)}{dx}\Delta x^{(\nu)}$$

6. Solve for $\Delta x^{(\nu)} = -\left[\frac{df\left(x^{(\nu)}\right)}{dx}\right]^{-1} f\left(x^{(\nu)}\right)$
7. Solve for new estimate of $\hat{x}$

$$x^{(\nu+1)} = x^{(\nu)} + \Delta x^{(\nu)}$$

Solve $x - \sqrt{x} - 1 = 0$

Guess $x^{(0)} = 1$

$$f(x) = x - \sqrt{x} - 1$$

$$\frac{df(x)}{dt} = 1 - \frac{1}{2}\, x^{-\frac{1}{2}}$$

$$\Delta x^{(\nu)} = -\left[\frac{df\left(x^{(\nu)}\right)}{dx}\right]^{-1} f\left(x^{(\nu)}\right)$$

$$= -\left(1 - \frac{1}{2}\, x^{(\nu)^{-\frac{1}{2}}}\right)^{-1} \left(x^{(\nu)} - \sqrt{x^{(\nu)}} - 1\right)$$

$$x^{(\nu+1)} = x^{(\nu)} + \Delta x^{(\nu)}$$

$$x^{(\nu+1)} = x^{(\nu)} - \left(1 - \frac{1}{2}\, x^{(\nu)^{-\frac{1}{2}}}\right)^{-1} \left(x^{(\nu)} - \sqrt{x^{(\nu)}} - 1\right)$$

Newton-Raphson Method, example 2

ν	$x^{(\nu)}$
0	1
1	3.0000000000000000
2	2.6233096782319105
3	2.6180351744324595
4	2.6180339887499550
5	2.6180339887498950
6	2.6180339887498950

Solve $x - \sqrt{x} - 1 = 0$

Guess $x^{(0)} = 1$

$f(x) = x - \sqrt{x} - 1$

$\frac{df(x)}{dt} = 1 - \frac{1}{2}\, x^{-\frac{1}{2}}$

$$\Delta x^{(\nu)} = -\left[\frac{df\left(x^{(\nu)}\right)}{dx}\right]^{-1} f\left(x^{(\nu)}\right)$$

$$= -\left(1 - \frac{1}{2}\, x^{(\nu)^{-\frac{1}{2}}}\right)^{-1} \left(x^{(\nu)} - \sqrt{x^{(\nu)}} - 1\right)$$

$$x^{(\nu+1)} = x^{(\nu)} + \Delta x^{(\nu)}$$

$$x^{(\nu+1)}$$

$$= x^{(\nu)} - \left(1 - \frac{1}{2}\, x^{(\nu)^{-\frac{1}{2}}}\right)^{-1} \left(x^{(\nu)} - \sqrt{x^{(\nu)}} - 1\right)$$

Stopping Criteria

- How many iterations do we go?
- Choose a tolerance ϵ
- Stop when $\left|\Delta x^{(\nu)}\right| < \epsilon$, where $\Delta x^{(\nu)} = x^{(\nu+1)} - x^{(\nu)}$
- If x is a vector (multi-variable problems), need to take the norm
- $\left\|\Delta x^{(\nu)}\right\|_j < \epsilon$
- Two common norms are the Euclidian & infinity

$$\left\|\Delta x^{(\nu)}\right\|_2 = \sqrt{\sum_{i=1}^{n} \Delta x_i^2} \qquad \left\|\Delta x^{(\nu)}\right\|_\infty = \max_{\mathrm{i}} |\Delta x_i|$$

Compare Gauss and Newton-Raphson

- Gauss
  - Relatively fast iterations, easy to calculate and program
  - For power flow, requires programming complex numbers
  - Convergence can be slower and unreliable
- Newton-Raphson
  - More complicated because it requires calculating the derivative
  - Much better convergence: when close to solution, the error decreases quite quickly—method has quadratic convergence

Analytical solution: $1.5 + 0.5\sqrt{5}$ =2.618033988749895

ν	$x^{(\nu)}$ (Gauss)	$x^{(\nu)}$ (Newton Raphson)
0	1	1
1	2	3.000000000000000
2	2.41421	2.6233096782319105
3	2.55377	2.6180351744324595
4	2.59805	2.6180339887499550
5	2.61185	2.6180339887498950
6	2.61612	2.6180339887498950
7	2.61744	
8	2.61785	
9	2.61798	
10	2.61802	
11	2.61803	

After 4 iterations, error of Gauss is 0.02,

Error of Newton-Raphson is 6×10^{-14}

Multi-Variable Newton-Raphson

- Next, we generalize to the case where x is an n-dimension vector, and f(x)=0 is an n-dimensional function

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_n(\mathbf{x}) \end{bmatrix}$$

- Again, define $\Delta\mathbf{x} = \hat{\mathbf{x}} - \mathbf{x}$
- Taylor series for $\mathbf{f}(\mathbf{x})$

$$\mathrm{f}_1(\hat{\mathbf{x}}) = \mathrm{f}_1(\mathbf{x}) + \frac{\partial \mathrm{f}_1(\mathbf{x})}{\partial x_1}\Delta x_1 + \frac{\partial \mathrm{f}_1(\mathbf{x})}{\partial x_2}\Delta x_2 + \cdots \frac{\partial \mathrm{f}_1(\mathbf{x})}{\partial x_n}\Delta x_n + \text{higher order terms}$$

$$\vdots$$

$$\mathrm{f}_\mathrm{n}(\hat{\mathbf{x}}) = \mathrm{f}_\mathrm{n}(\mathbf{x}) + \frac{\partial \mathrm{f}_\mathrm{n}(\mathbf{x})}{\partial x_1}\Delta x_1 + \frac{\partial \mathrm{f}_\mathrm{n}(\mathbf{x})}{\partial x_2}\Delta x_2 + \cdots \frac{\partial \mathrm{f}_\mathrm{n}(\mathbf{x})}{\partial x_n}\Delta x_n + \text{higher order terms}$$

Jacobian Matrix

- We can write the truncated Taylor series in Matrix form as the Jacobian matrix

$$\mathbf{J}(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1(\mathbf{x})}{\partial x_1} & \frac{\partial f_1(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_1(\mathbf{x})}{\partial x_n} \\ \frac{\partial f_2(\mathbf{x})}{\partial x_1} & \frac{\partial f_2(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_2(\mathbf{x})}{\partial x_n} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{\partial f_n(\mathbf{x})}{\partial x_1} & \frac{\partial f_n(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_n(\mathbf{x})}{\partial x_n} \end{bmatrix}$$

- $\mathbf{f}(\hat{\mathbf{x}}) = \mathbf{f}\left(\mathbf{x}^{(\nu)}\right) + \mathbf{J}(\mathbf{x}) \cdot \mathbf{\Delta x}^{(\nu)} = \mathbf{0}$
- So, $\mathbf{\Delta x}^{(\nu)} = -\mathbf{J}(\mathbf{x})^{-1} \cdot \mathbf{f}\left(\mathbf{x}^{(\nu)}\right)$
- And so $\mathbf{x}^{(\nu+1)} = \mathbf{x}^{(\nu)} - \mathbf{J}(\mathbf{x})^{-1} \cdot \mathbf{f}\left(\mathbf{x}^{(\nu)}\right)$

Multi-Variable Newton-Raphson Example

Solve for x_1 and x_2 such that

$$f_1(x_1, x_2) = 2x_1^2 + x_2^2 - 8 = 0$$

$$f_2(x_1, x_2) = x_1^2 - x_2^2 + x_1 x_2 - 4 = 0$$

Multi-Variable Newton-Raphson Example 2

Solve for x_1 and x_2 such that

$$f_1(x_1, x_2) = 2x_1^2 + x_2^2 - 8 = 0$$

$$f_2(x_1, x_2) = x_1^2 - x_2^2 + x_1 x_2 - 4 = 0$$

First, symbolically determine the Jacobian

$$\mathbf{J}(\mathbf{x}) = \begin{bmatrix} \frac{\partial \mathrm{f}_1(\mathbf{x})}{\partial x_1} & \frac{\partial \mathrm{f}_1(\mathbf{x})}{\partial x_2} \\ \frac{\partial \mathrm{f}_2(\mathbf{x})}{\partial x_1} & \frac{\partial \mathrm{f}_2(\mathbf{x})}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 4x_1 & 2x_2 \\ 2x_1 + x_2 & x_1 - 2x_2 \end{bmatrix}$$

Then

$$\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = -\begin{bmatrix} 4x_1 & 2x_2 \\ 2x_1 + x_2 & x_1 - 2x_2 \end{bmatrix}^{-1} \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \end{bmatrix}$$

Arbitrarily guess $\mathbf{x}^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}^{-1} \begin{bmatrix} -5 \\ -3 \end{bmatrix} = \begin{bmatrix} 2.1 \\ 1.3 \end{bmatrix}$$

$$\mathbf{x}^{(2)} = \begin{bmatrix} 2.1 \\ 1.3 \end{bmatrix} - \begin{bmatrix} 8.40 & 2.60 \\ 5.50 & -0.50 \end{bmatrix}^{-1} \begin{bmatrix} 2.51 \\ 1.45 \end{bmatrix} = \begin{bmatrix} 1.8284 \\ 1.2122 \end{bmatrix}$$

Each iteration we check $\|\mathbf{f}(\mathbf{x})\|$ to see if it is below our specified tolerance ε

$$\mathbf{f}(\mathbf{x}^{(2)}) = \begin{bmatrix} 0.1556 \\ 0.0900 \end{bmatrix}$$

If ε = 0.2 then we would be done. Otherwise, we'd continue iterating.

Power Balance Equations

Current injection is given by the Y-bus formulation we have from before

$$\mathbf{I} = \mathbf{Y_{bus}} \mathbf{V}$$

For each bus *i* in the system, the *power injection* (generation – load) is thus

$$\bar{S}_i = \bar{V}_i \bar{I}_i^* = \bar{V}_i \left(\sum_{k=1}^{n} \bar{Y}_{ik} \bar{V}_k \right) = \bar{V}_i \sum_{k=1}^{n} \bar{Y}_{ik}^* \bar{V}_k^*$$

This is a complex equation. Next we want to convert it to a set of real equations for P_i and Q_i

Power Balance Equations, cont.

First, define $\bar{Y}_{ik}^{*} = g_{ik} + jb_k$ and $\bar{V}_i = V_i \angle \theta_i = V_i(\cos\theta_i + j\sin\theta_i)$

And use $\theta_{ij} = \theta_i - \theta_j$

$$\bar{S}_i = P_i + jQ_i = \bar{V}_i \sum_{k=1}^{n} \bar{Y}_{ik}^{*} \bar{V}_k^{*} = \sum_{k=1}^{n} |V_i||V_k| \ e^{j\theta_{ik}}(g_{ik} - jb_{ik})$$

$$= \sum_{k=1}^{n} |V_i||V_k| \ (\cos\theta_{ik} + j\sin\theta_{ik})(g_{ik} - jb_{ik})$$

Resolve into real and imaginary parts:

$$\mathrm{P_i} = \sum_{k=1}^{n} |V_i||V_k| (g_{ik}\cos\theta_{ik} + b_{ik}\sin\theta_{ik}) = P_{Gi} - P_{Di}$$

$$\mathrm{Q_i} = \sum_{k=1}^{n} |V_i||V_k| (g_{ik}\sin\theta_{ik} - b_{ik}\cos\theta_{ik}) = Q_{Gi} - Q_{Di}$$

Power balance equations

Power Flow Problem Variables

- These two equations must be satisfied for all buses in the system
- Total of 2n equations
- Parameters g and b are known from the Y-bus
- Each bus has four variables:
  - Bus voltage magnitude V_i
  - Bus voltage angle θ_i
  - Bus real power injection P_i
  - Bus reactive power injection Q_i
- So in general, we need to pre-specify 2 additional variables at each bus before we can solve

$$\mathrm{P_i} = \sum_{k=1}^{n} |V_i||V_k| \left(g_{ik} \cos\theta_{ik} + b_{ik} \sin\theta_{ik}\right) = P_{Gi} - P_{Di}$$

$$\mathrm{Q_i} = \sum_{k=1}^{n} |V_i||V_k| \left(g_{ik} \sin\theta_{ik} - b_{ik} \cos\theta_{ik}\right) = Q_{Gi} - Q_{Di}$$

Power Flow Bus Types

Each bus has one of three basic types, and for each one there are two known variables and two unknown variables:

- Load (PQ) bus
  - Assume P/Q are fixed at most buses in the system
  - Known: P and Q, Unknown: V and θ
- Slack bus
  - Only one in the whole system
  - Usually assigned to a large generator bus
  - Known: V and θ, Unknown: P and Q
  - Provides angle reference ($\theta = 0$) and picks up the slack P
- Generator (PV) bus
  - Other generators (5-10% of buses in a large system)
  - Assume generator is controlling power output and voltage
  - Known: V and P, Unknown: θ and Q