ECEN 667
Power System Stability

Lecture 18: Stability Signal Analysis, Part 1

Prof. Adam Birchfield
Dept. of Electrical and Computer Engineering
Texas A&M University
abirchfield@tamu.edu

Announcements

- HW #4 is on the website, due Nov 6th at 8 AM.
- Read the following paper by Nov 11th :
  - S. Maslennikov and E. Litvinov, "ISO New England Experience in Locating the Source of Oscillations Online," in *IEEE Transactions on Power Systems*, vol. 36, no. 1, pp. 495-503, Jan. 2021
- HW #5 is on the website, due Nov 20th at 8 AM.
- Review the slides and PowerWorld examples
- **Exam 2 will be Tuesday, December 2nd, 2025**

Oscillations

- An oscillation is just a repetitive motion that can be either undamped, positively damped (decaying with time) or negatively damped (growing with time)
- If the oscillation can be written as a sinusoid then

$$e^{\alpha t}\left(a\cos\left(\omega t\right)+b\sin\left(\omega t\right)\right)=e^{\alpha t}C\cos\left(\omega t+\theta\right)$$

$$\text{where } C=\sqrt{A^2+B^2} \text{ and } \theta=\tan\left(\frac{-b}{a}\right)$$

- The damping ratio is

$$\xi=\frac{-\alpha}{\sqrt{\alpha^2+\omega^2}}$$

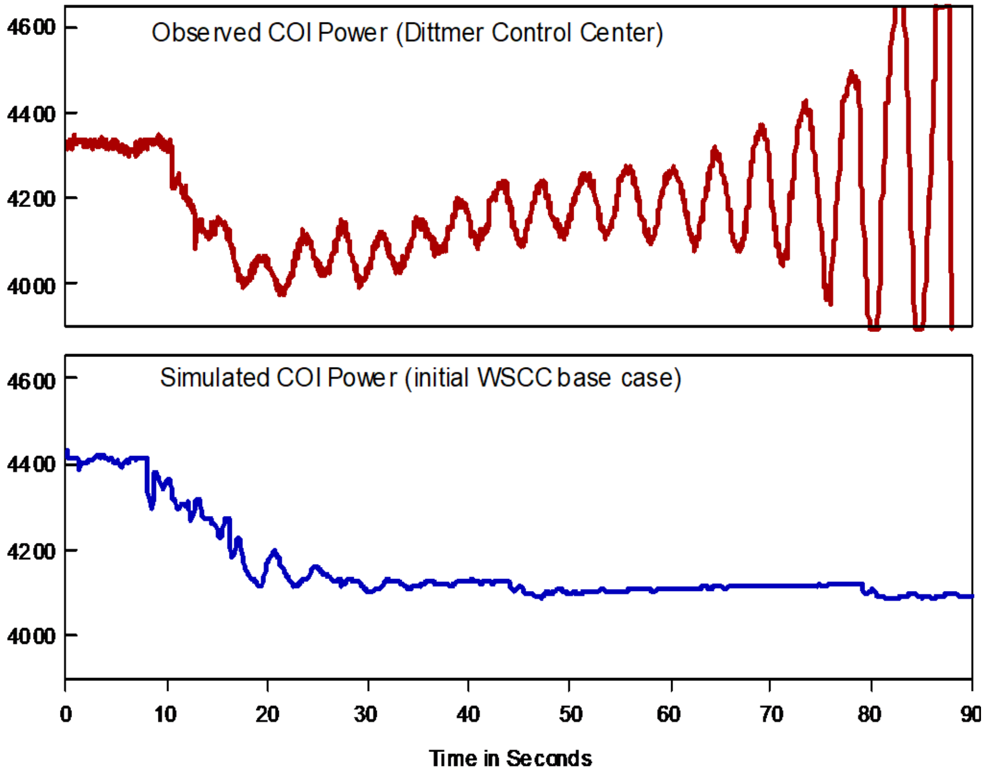
The percent damping is just the damping ratio multiplied by 100; goal is sufficiently positive damping

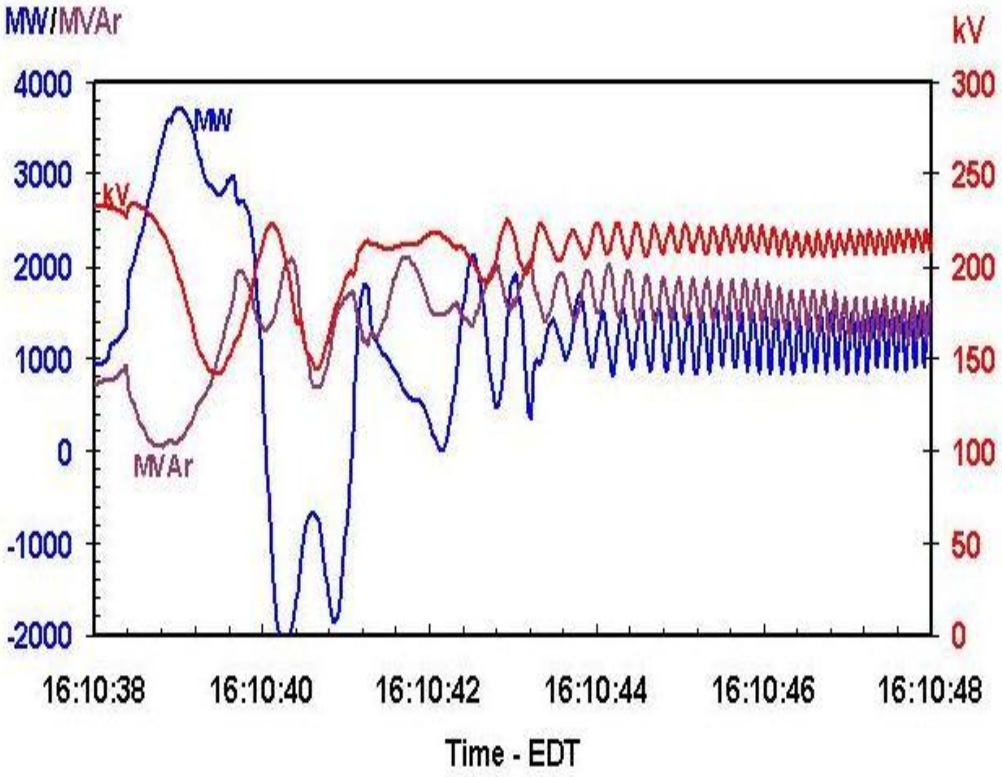
Power System Oscillations

- Power systems can experience a wide range of oscillations, ranging from highly damped and high frequency switching transients to sustained low frequency (< 2 Hz) inter-area oscillations affecting an entire interconnect
- Types of oscillations include
  - Transients: Usually high frequency and highly damped
  - Local plant: Usually from 1 to 5 Hz
  - Inter-area oscillations: From 0.15 to 1 Hz
  - Slower dynamics: Such as AGC, less than 0.15 Hz
  - Subsynchronous resonance: 10 to 50 Hz (less than synchronous)

Example Oscillations

- The left graph shows an oscillation that was observed during a 1996 WECC Blackout, the right from the 8/14/2003 blackout





References

- For the 1996 WECC blackout, more information is available at
  - www.nerc.com/pa/rrm/ea/System%20Disturbance%20Reports%20DL/1996SystemDisturbance.pdf; the July 2, 1996 event was caused by a tree contact
- Charlie Concordia wrote a paper on electric grid oscillations in 1938
  - C. Concordia, S. B. Crary, J.M. Lyons, Stability Characteristics of Turbine Generators, *AIEE Transactions*, vol. 57, pp. 732-744, 1938.
- There is a 2021 NERC document on oscillations at www.nerc.com/comm/PC/SMSResourcesDocuments/Interconnection_Oscillation_Analysis.pdf
  - Also see T.J. Overbye, S. Kunkolienkar, F. Safdarian, A. Birchfield, “On the Existence of Dominant Inter-Area Oscillation Modes in the North American Eastern Interconnect Stability Simulations”, 57th Hawaii International Conference on System Sciences, Honolulu, HI, January 2024

Modes

- A mode is a concept from linear system analysis
  - Electric grids certainly are not linear, but usually their response to small disturbances is approximated as linear
- A mode corresponds to one of the eigenvalues of the response or, for oscillations, a complex pair of eigenvalues
- A mode has a frequency and damping; all parts of the system oscillate with this pattern
- The mode shape tells how parts of the system participate in the mode
- There can be multiple modes in a system; power systems can have many modes

Causes of Power System Oscillations

- The response of a simple system can be divided into its natural response versus its forced response
  - The natural response tells how the system will response to an initial disturbance without any additional (external) influences; this response shows the system's modes
  - A forced response is associated with an external disturbance; if the external disturbance is periodic then the system will oscillate at least partially at this frequency
  - Often forced oscillations are due to control failures
- Resonance occurs when a forced response is at a similar frequency to one of the system's modes
- An power system can experience both types of oscillations

Forced Oscillations in WECC (from [1])

- Summer 2013 24 hour data: 0.37 Hz oscillations observed for several hours. Confirmed to be forced oscillations at a hydro plant from vortex effect.
- 2014 data: Another 0.5 Hz oscillation also observed. Source points to hydro unit as well. And 0.7 Hz. And 1.12 Hz. And 2 Hz.
- Resonance possible when system modes are poorly damped and close to the forcing function. Resonance can be observed in model simulations.

1. M. Venkatasubramanian, “Oscillation Monitoring System”, June 2015 http://www.energy.gov/sites/prod/files/2015/07/f24/3.%20Mani%20Oscillation%20Monitoring.pdf

Observing Modes and Damping

- With the advent of wide-scale PMU deployments, the modes and damping can be observed two ways
  - Event (ringdown) analysis – this requires an event
  - Ambient noise analysis – always available, but not as distinct

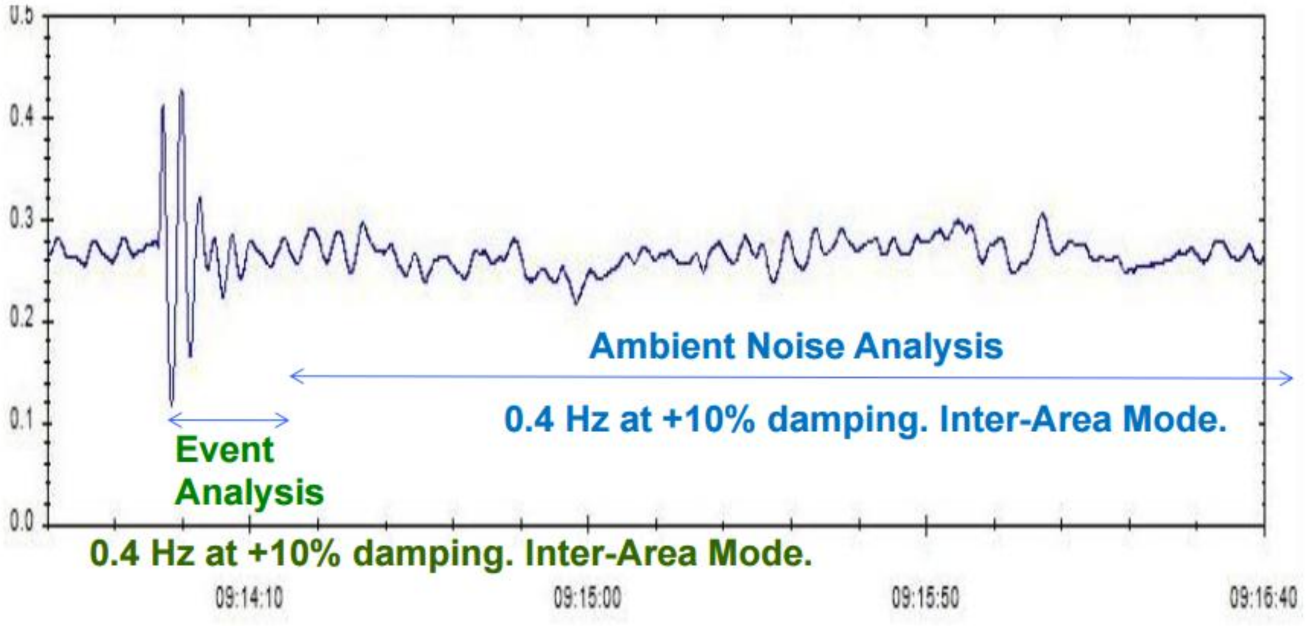


Image Source: M. Venkatasubramanian, “Oscillation Monitoring System”, June 2015
http://www.energy.gov/sites/prod/files/2015/07/f24/3.%20Mani%20Oscillation%20Monitoring.pdf

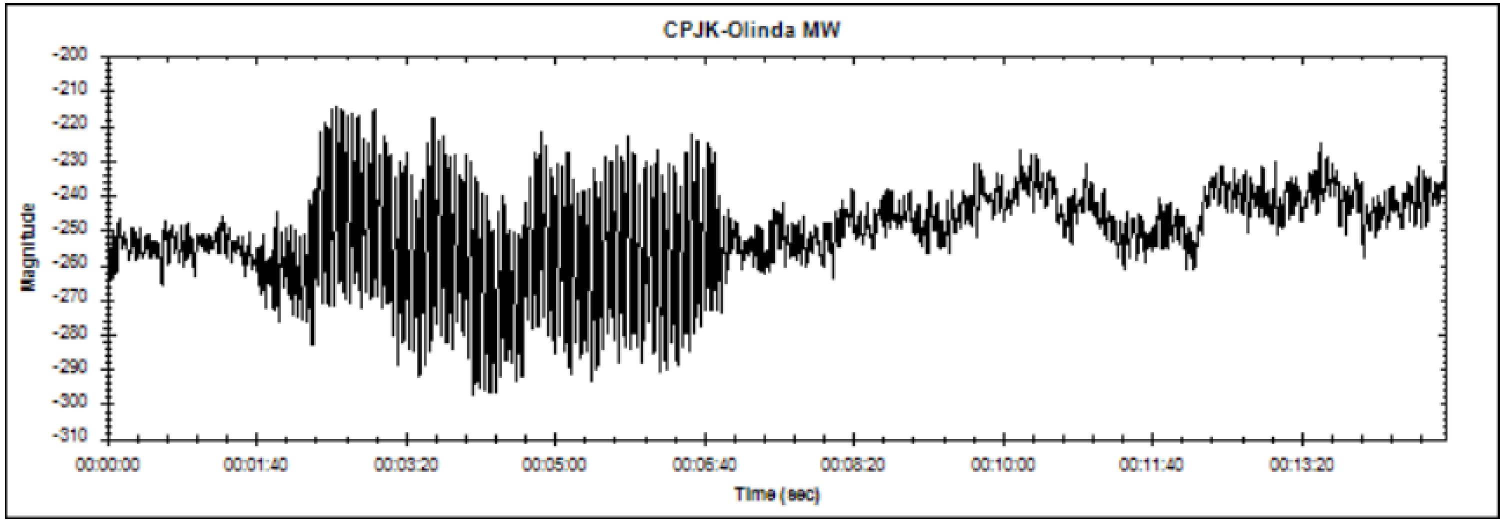
Resonance with Interarea Mode [1]

- Resonance effect high when:
  - Forced oscillation frequency near system mode frequency
  - System mode poorly damped
  - Forced oscillation location near the two distant ends of mode
- Resonance effect medium when
  - Some conditions hold
- Resonance effect small when
  - None of the conditions holds

1. M. Venkatasubramanian, “Oscillation Monitoring System”, June 2015
http://www.energy.gov/sites/prod/files/2015/07/f24/3.%20Mani%20Oscillation%20Monitoring.pdf

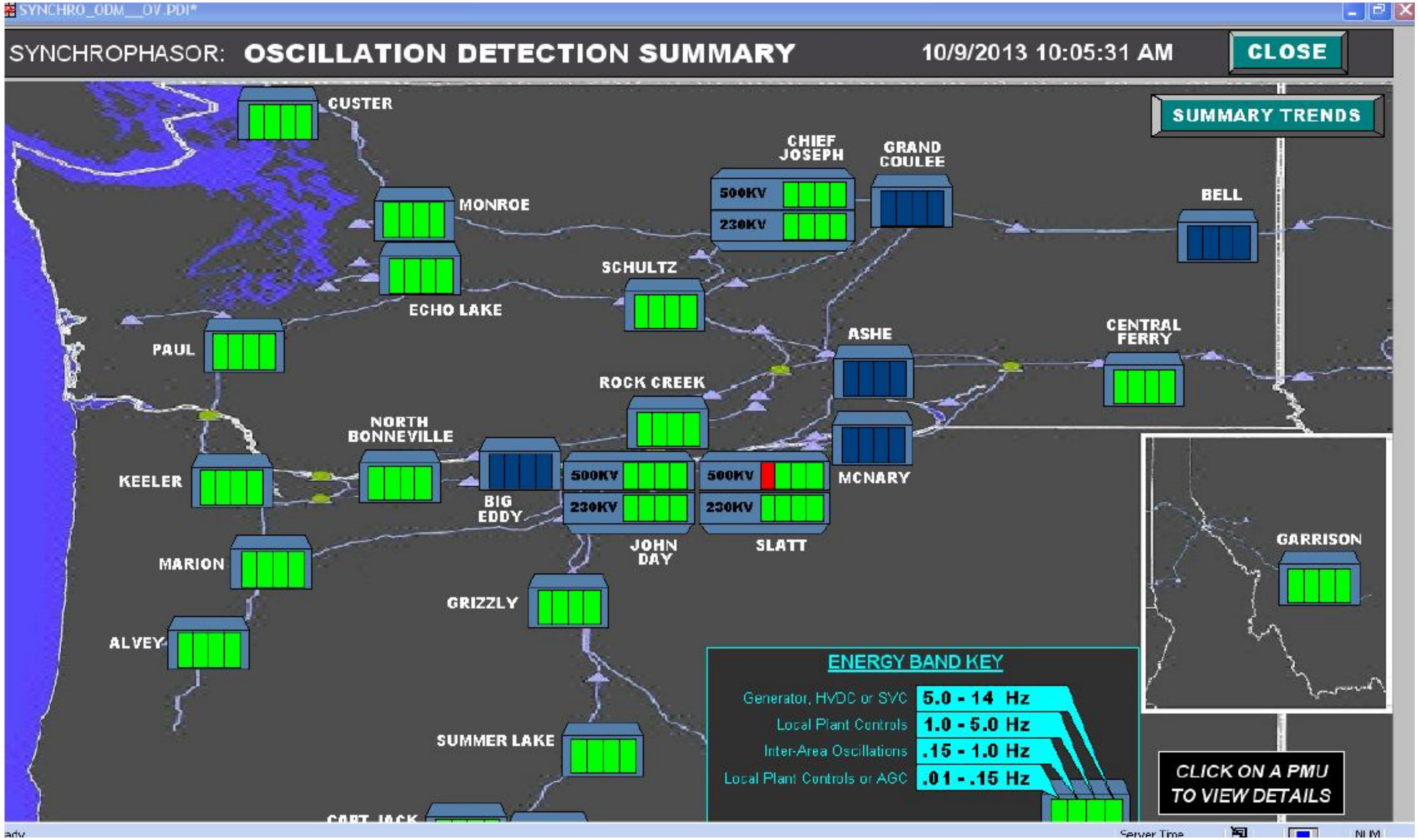
Medium Resonance on 11/29/2005

- 20 MW 0.26 Hz Forced Oscillation in Alberta Canada
- 200 MW Oscillations on California-Oregon Inter-tie
- System mode 0.27 Hz at 8% damping
- Two out of the three conditions were true.



1. M. Venkatasubramanian, “Oscillation Monitoring System”, June 2015 http://www.energy.gov/sites/prod/files/2015/07/f24/3.%20Mani%20Oscillation%20Monitoring.pdf

An Online Oscillation Detection Tool



ENERGY BAND KEY

Generator, HVDC or SVC	5.0 - 14 Hz
Local Plant Controls	1.0 - 5.0 Hz
Inter-Area Oscillations	.15 - 1.0 Hz
Local Plant Controls or AGC	.01 - .15 Hz

Image source: WECC Joint Synchronized Information Subcommittee Report, October 2013

Small Signal Analysis and Measurement-Based Modal Analysis

- Small signal analysis has been used for decades to determine power system frequency response
  - It is a model-based approach that considers the properties of a power system, linearized about an operating point
- Measurement-based modal analysis determines the observed dynamic properties of a system
  - Input can either be measurements from devices (such as PMUs) or dynamic simulation results
  - The same approach can be used regardless of the measurement source
- Focus in this section is on the measurement-based approach

Ring-down Modal Analysis

- Ring-down analysis seeks to determine the frequency and damping of key power system modes following some disturbance
- There are several different techniques, with the Prony approach the oldest (from 1795); introduced into power in 1990 by Hauer, Demeure and Scharf
- Regardless of technique, the goal is to represent the response of a sampled signal as a set of exponentially damped sinusoidals (modes)

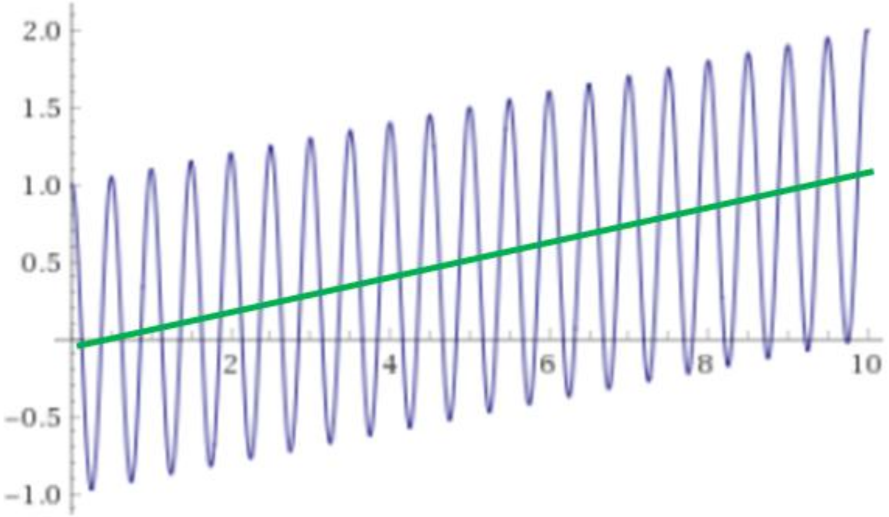
$$y(t) = \sum_{i=1}^{q} A_i e^{\sigma_i t} \cos(\omega_i t + \phi_i)$$

$$\text{Damping } (\%) = \frac{-\alpha_i}{\sqrt{\alpha_i^2 + \omega_i^2}} \times 100$$

Goal: Extracting Modes from the Signals

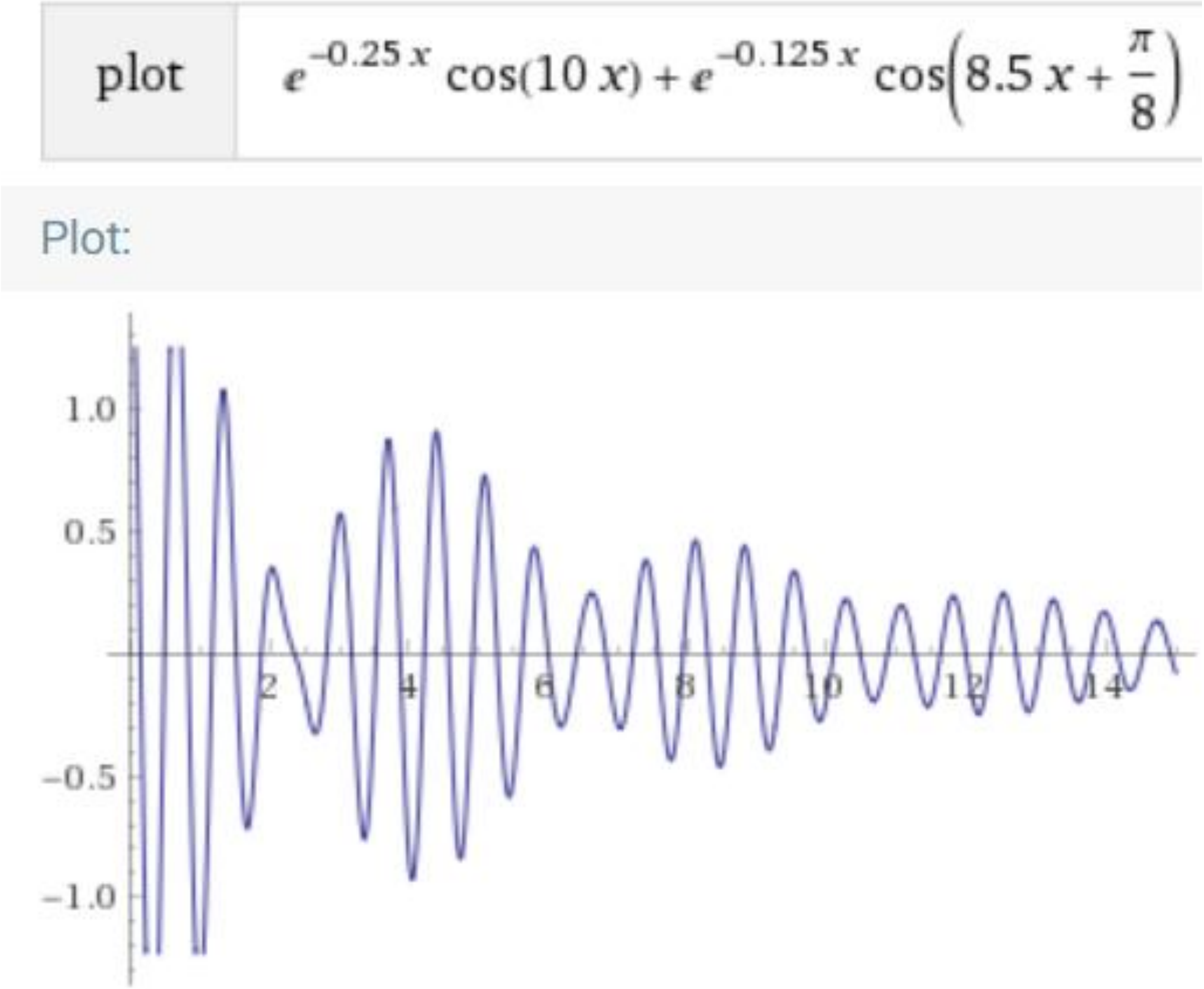
- The goal is to gain information about the electric grid by extracting modal information from its signals
  - The frequency and damping of the modes is key
- The premise is we'll be able to reproduce a complex signal, over a period of time, as a set a of sinusoidal modes
  - We'll also allow for linear detrending

$$0.1t + \cos(2\pi 2t)$$



Example: Summation of Two Damped Exponentials

- This example was created by going from the modes to a signal
- We'll be going in the opposite direction (i.e., from a measured signal to the modes)



plot $e^{-0.25x}\cos(10x)+e^{-0.125x}\cos\left(8.5x+\frac{\pi}{8}\right)$

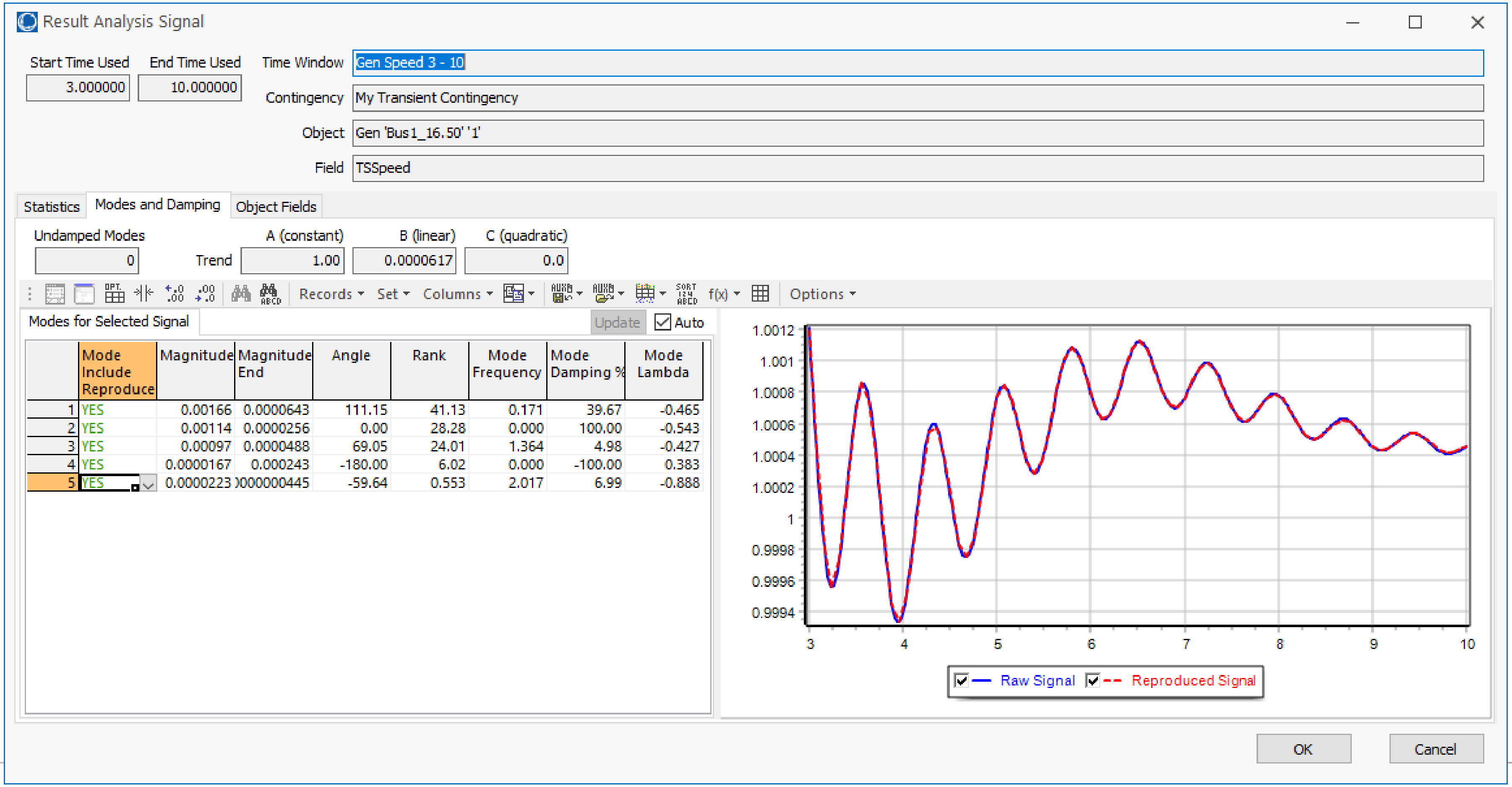
Plot:

Some Reasonable Expectations

- **Verifiable** to show how well the modes match the original signal(s)
  - We'll show this
- **Flexible** to handle between one and many signals
  - We'll go up to simultaneously considering 40,000 signals
- **Fast**
  - What is presented will be, with a discussion of the computational scaling
- **Easy to use**
  - This is software implementation specific; results shown here were done using the modal analysis tool integrated into PowerWorld Simulator (version 22)

Example: One Signal

- This could be any signal; image shows the result of the original signal (blue) and the reproduced signal (red)



Result Analysis Signal

Start Time Used: 3.000000
End Time Used: 10.000000
Time Window: Gen Speed 3 - 10
Contingency: My Transient Contingency
Object: Gen 'Bus1_16.50' '1'
Field: TSSpeed

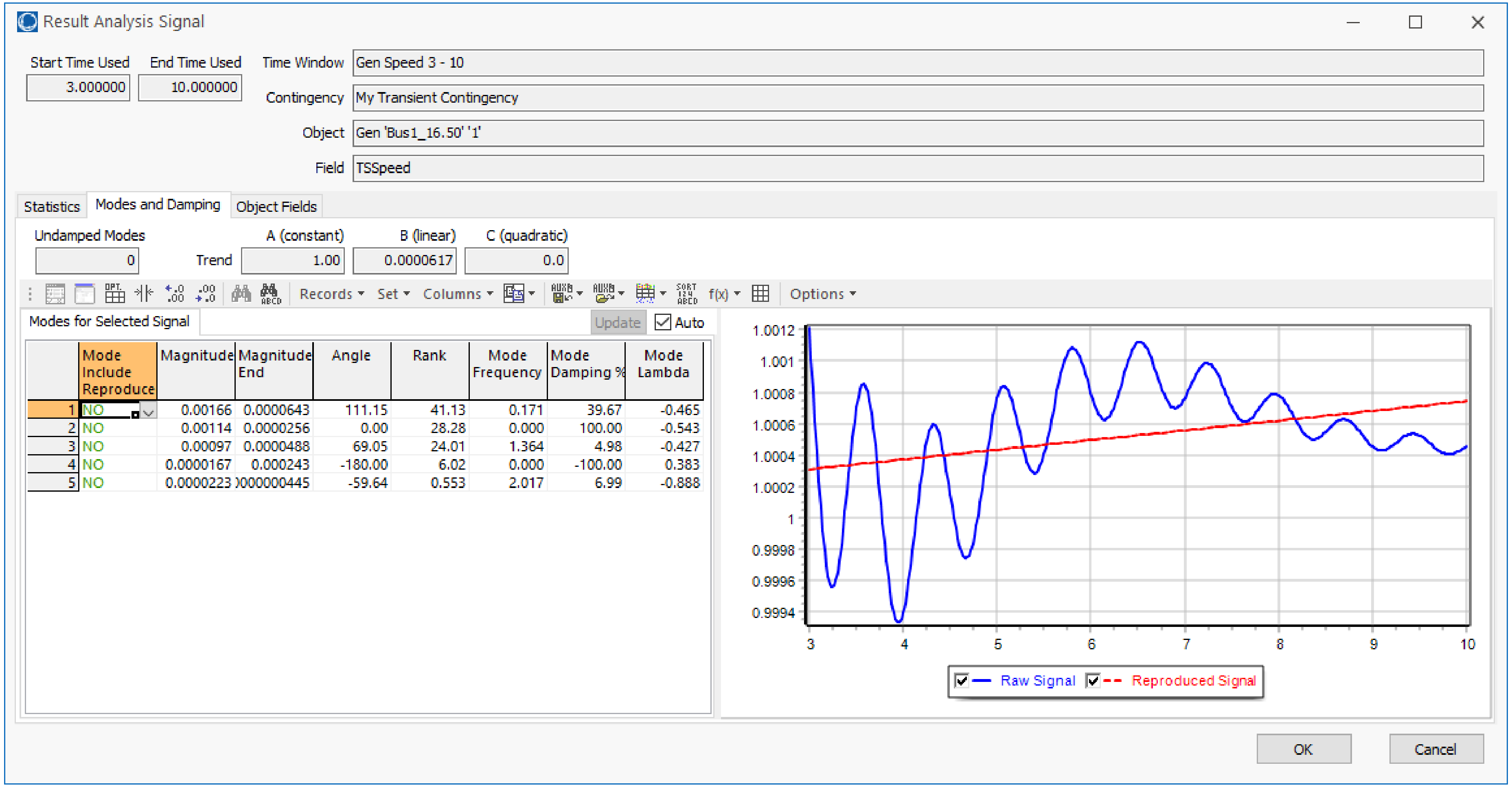
Statistics | Modes and Damping | Object Fields

Undamped Modes: 0
Trend: A (constant) 1.00; B (linear) 0.0000617; C (quadratic) 0.0

Modes for Selected Signal

	Mode Include Reproduce	Magnitude	Magnitude End	Angle	Rank	Mode Frequency	Mode Damping %	Mode Lambda
1	YES	0.00166	0.0000643	111.15	41.13	0.171	39.67	-0.465
2	YES	0.00114	0.0000256	0.00	28.28	0.000	100.00	-0.543
3	YES	0.00097	0.0000488	69.05	24.01	1.364	4.98	-0.427
4	YES	0.0000167	0.000243	-180.00	6.02	0.000	-100.00	0.383
5	YES	0.0000223	000000445	-59.64	0.553	2.017	6.99	-0.888

Verification: Linear Trend Line Only



Result Analysis Signal

Start Time Used: 3.000000

End Time Used: 10.000000

Time Window: Gen Speed 3 - 10

Contingency: My Transient Contingency

Object: Gen 'Bus1_16.50' '1'

Field: TSSpeed

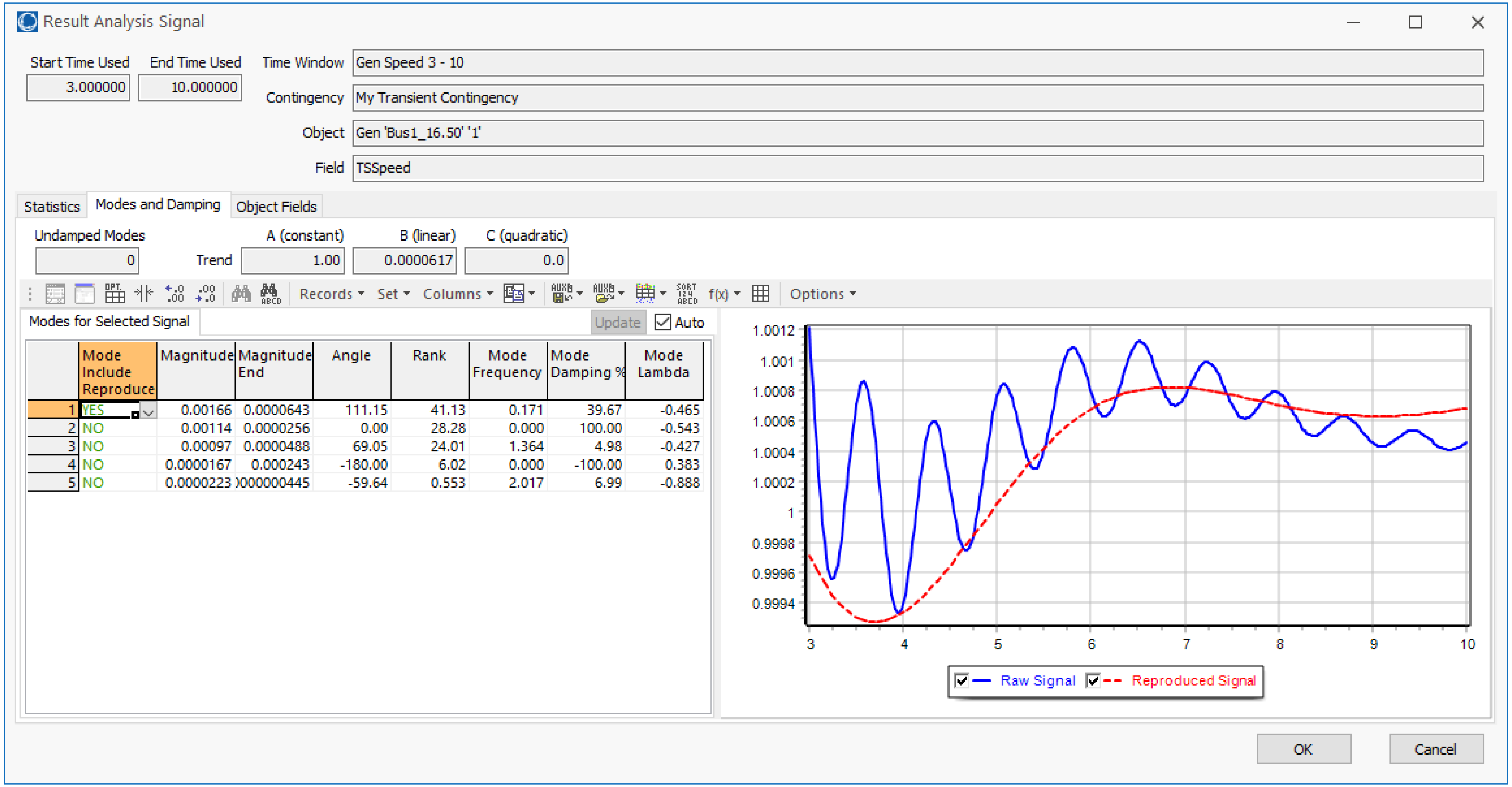
Undamped Modes: 0

	A (constant)	B (linear)	C (quadratic)
Trend	1.00	0.0000617	0.0

Modes for Selected Signal

	Mode Include Reproduce	Magnitude	Magnitude End	Angle	Rank	Mode Frequency	Mode Damping %	Mode Lambda
1	NO	0.00166	0.0000643	111.15	41.13	0.171	39.67	-0.465
2	NO	0.00114	0.0000256	0.00	28.28	0.000	100.00	-0.543
3	NO	0.00097	0.0000488	69.05	24.01	1.364	4.98	-0.427
4	NO	0.0000167	0.000243	-180.00	6.02	0.000	-100.00	0.383
5	NO	0.0000223	)000000445	-59.64	0.553	2.017	6.99	-0.888

Verification: Linear Trend Line + One Mode



Start Time Used: 3.000000

End Time Used: 10.000000

Time Window: Gen Speed 3 - 10

Contingency: My Transient Contingency

Object: Gen 'Bus1_16.50' '1'

Field: TSSpeed

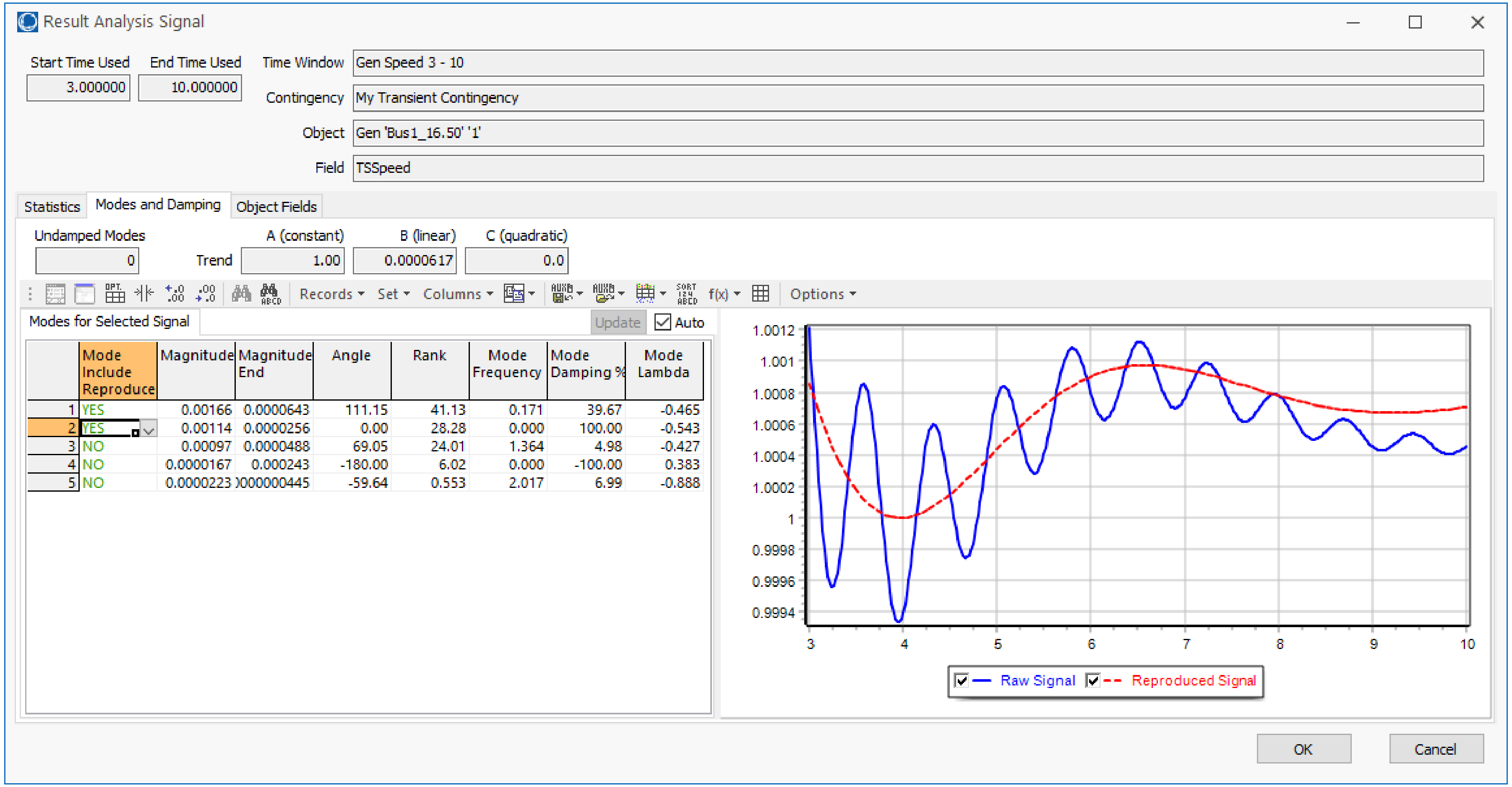
Undamped Modes: 0

Trend: A (constant) 1.00, B (linear) 0.0000617, C (quadratic) 0.0

Modes for Selected Signal

	Mode Include Reproduce	Magnitude	Magnitude End	Angle	Rank	Mode Frequency	Mode Damping %	Mode Lambda
1	YES	0.00166	0.0000643	111.15	41.13	0.171	39.67	-0.465
2	NO	0.00114	0.0000256	0.00	28.28	0.000	100.00	-0.543
3	NO	0.00097	0.0000488	69.05	24.01	1.364	4.98	-0.427
4	NO	0.0000167	0.000243	-180.00	6.02	0.000	-100.00	0.383
5	NO	0.0000223	)000000445	-59.64	0.553	2.017	6.99	-0.888

Verification: Linear Trend Line + Two Modes



Start Time Used: 3.000000

End Time Used: 10.000000

Time Window: Gen Speed 3 - 10

Contingency: My Transient Contingency

Object: Gen 'Bus1_16.50' '1'

Field: TSSpeed

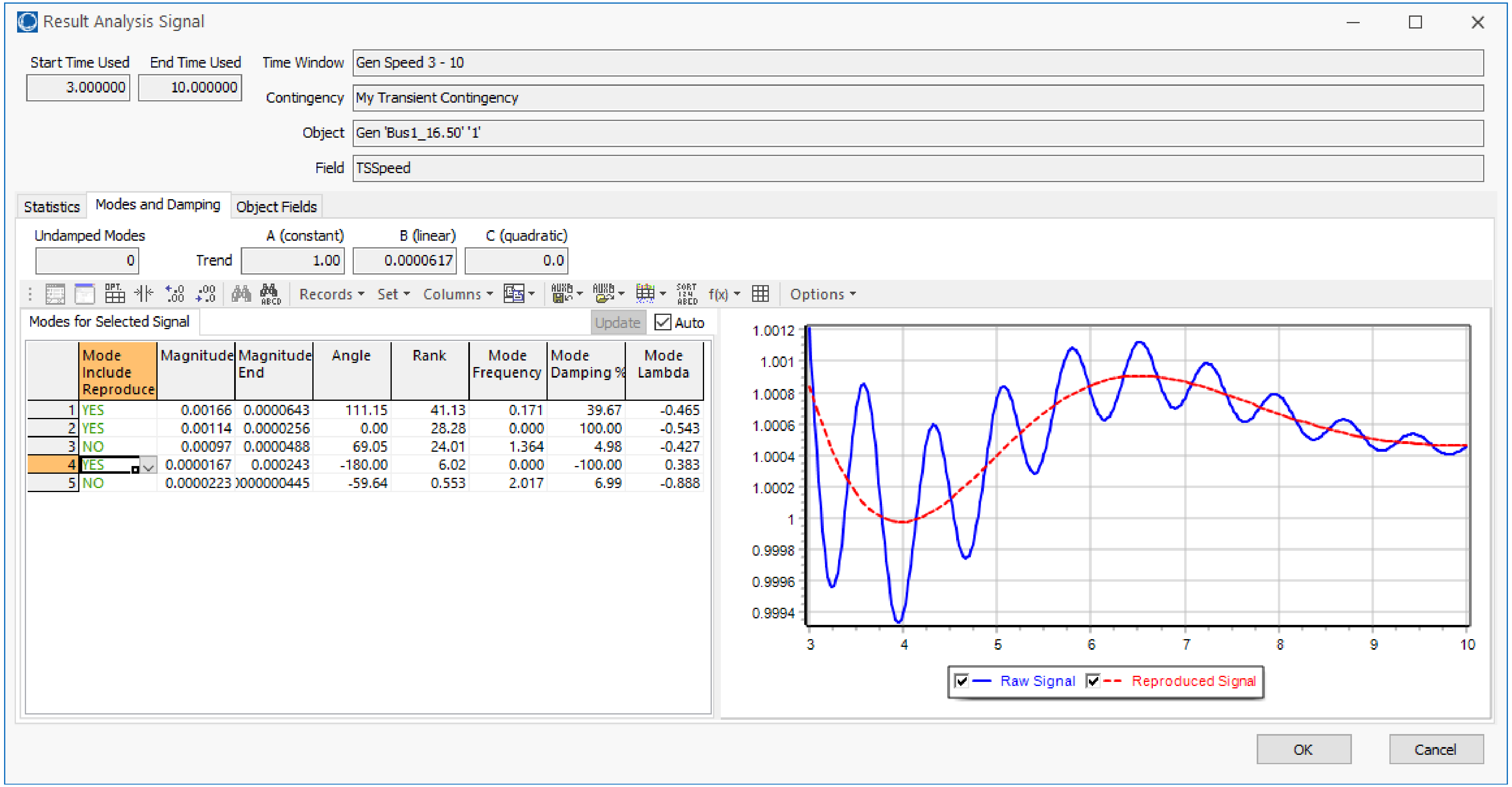
Undamped Modes: 0

Trend: A (constant) 1.00; B (linear) 0.0000617; C (quadratic) 0.0

Modes for Selected Signal

	Mode Include Reproduce	Magnitude	Magnitude End	Angle	Rank	Mode Frequency	Mode Damping %	Mode Lambda
1	YES	0.00166	0.0000643	111.15	41.13	0.171	39.67	-0.465
2	YES	0.00114	0.0000256	0.00	28.28	0.000	100.00	-0.543
3	NO	0.00097	0.0000488	69.05	24.01	1.364	4.98	-0.427
4	NO	0.0000167	0.000243	-180.00	6.02	0.000	-100.00	0.383
5	NO	0.0000223	)000000445	-59.64	0.553	2.017	6.99	-0.888

Verification: Linear Trend Line + Three Modes



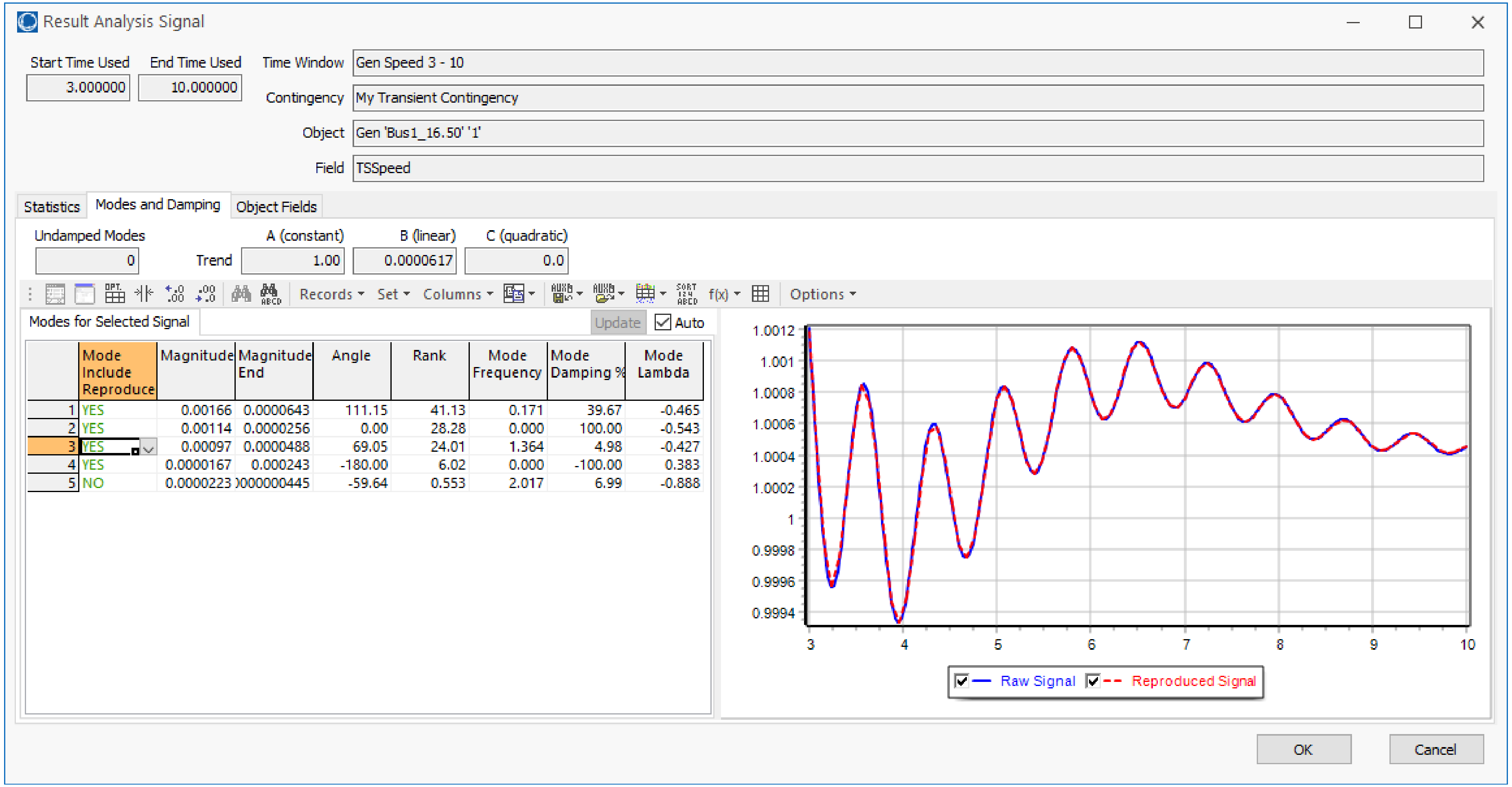
Start Time Used: 3.000000
End Time Used: 10.000000
Time Window: Gen Speed 3 - 10
Contingency: My Transient Contingency
Object: Gen 'Bus1_16.50' '1'
Field: TSSpeed

Undamped Modes: 0
Trend: A (constant) 1.00; B (linear) 0.0000617; C (quadratic) 0.0

Modes for Selected Signal

	Mode Include Reproduce	Magnitude	Magnitude End	Angle	Rank	Mode Frequency	Mode Damping %	Mode Lambda
1	YES	0.00166	0.0000643	111.15	41.13	0.171	39.67	-0.465
2	YES	0.00114	0.0000256	0.00	28.28	0.000	100.00	-0.543
3	NO	0.00097	0.0000488	69.05	24.01	1.364	4.98	-0.427
4	YES	0.0000167	0.000243	-180.00	6.02	0.000	-100.00	0.383
5	NO	0.0000223	)000000445	-59.64	0.553	2.017	6.99	-0.888

Verification: Linear Trend Line + Four Modes



Start Time Used: 3.000000

End Time Used: 10.000000

Time Window: Gen Speed 3 - 10

Contingency: My Transient Contingency

Object: Gen 'Bus1_16.50' '1'

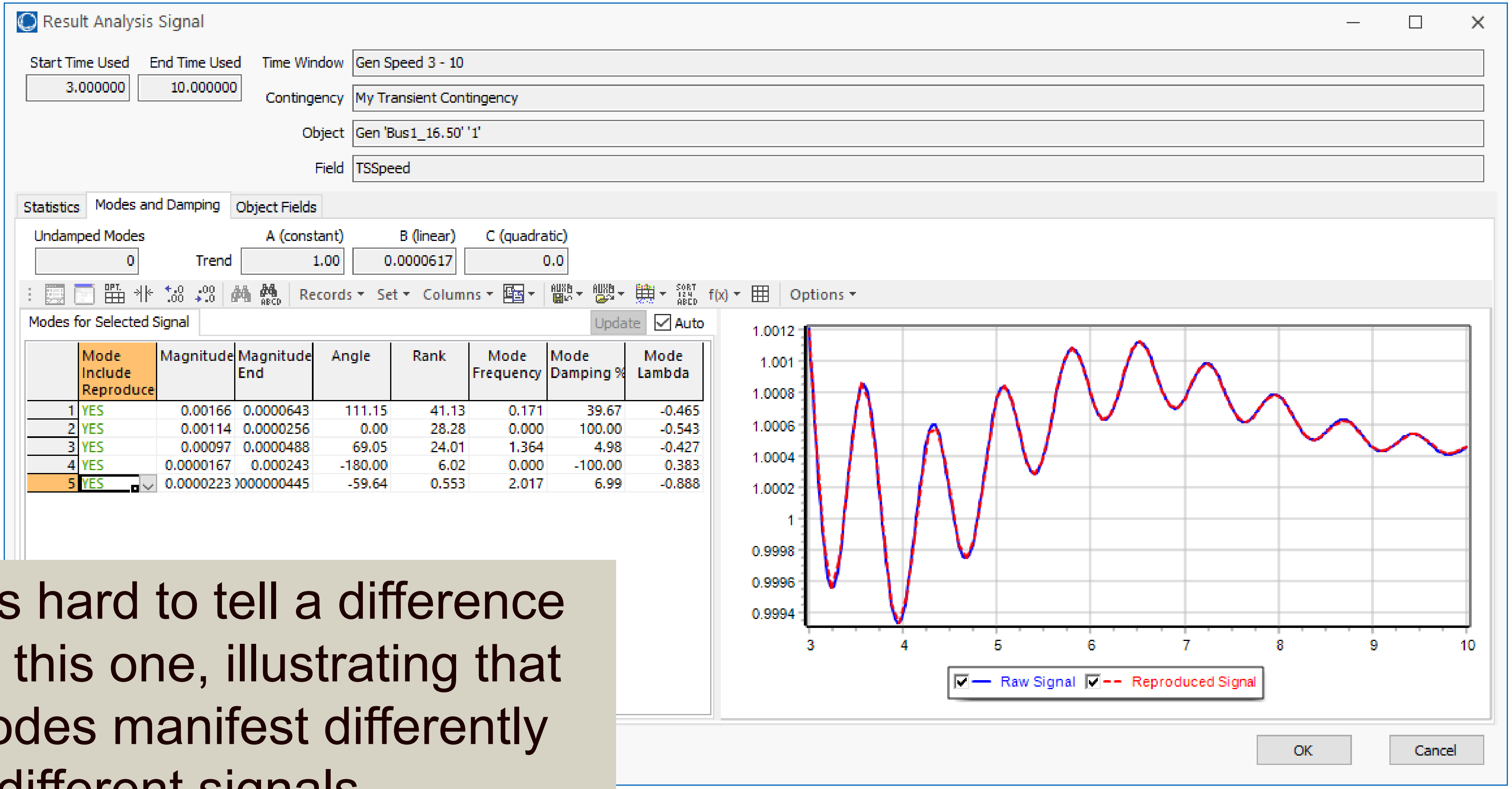
Field: TSSpeed

Undamped Modes: 0

	A (constant)	B (linear)	C (quadratic)
Trend	1.00	0.0000617	0.0

	Mode Include Reproduce	Magnitude	Magnitude End	Angle	Rank	Mode Frequency	Mode Damping %	Mode Lambda
1	YES	0.00166	0.0000643	111.15	41.13	0.171	39.67	-0.465
2	YES	0.00114	0.0000256	0.00	28.28	0.000	100.00	-0.543
3	YES	0.00097	0.0000488	69.05	24.01	1.364	4.98	-0.427
4	YES	0.0000167	0.000243	-180.00	6.02	0.000	-100.00	0.383
5	NO	0.0000223	000000445	-59.64	0.553	2.017	6.99	-0.888

Verification: Linear Trend Line + Five Modes



Result Analysis Signal

Start Time Used: 3.000000

End Time Used: 10.000000

Time Window: Gen Speed 3 - 10

Contingency: My Transient Contingency

Object: Gen 'Bus1_16.50' '1'

Field: TSSpeed

Undamped Modes: 0

Trend: A (constant) 1.00; B (linear) 0.0000617; C (quadratic) 0.0

Modes for Selected Signal

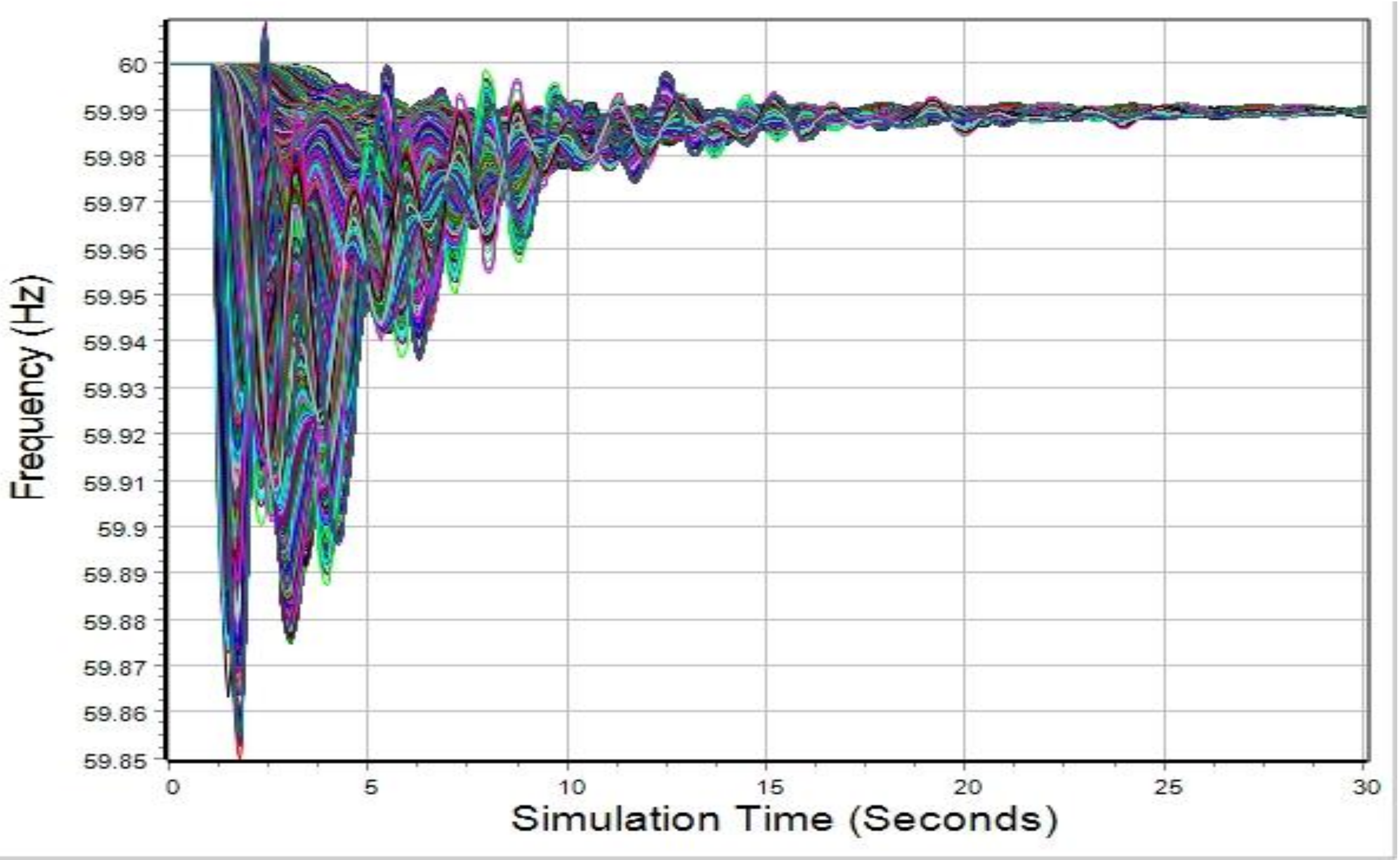
☑ Auto

	Mode Include Reproduce	Magnitude	Magnitude End	Angle	Rank	Mode Frequency	Mode Damping %	Mode Lambda
1	YES	0.00166	0.0000643	111.15	41.13	0.171	39.67	-0.465
2	YES	0.00114	0.0000256	0.00	28.28	0.000	100.00	-0.543
3	YES	0.00097	0.0000488	69.05	24.01	1.364	4.98	-0.427
4	YES	0.0000167	0.000243	-180.00	6.02	0.000	-100.00	0.383
5	YES	0.0000223	)000000445	-59.64	0.553	2.017	6.99	-0.888

It is hard to tell a difference on this one, illustrating that modes manifest differently in different signals

A Larger Example

- Applying the developed techniques to the response of all 43,400 substation frequencies from a 110,000-bus electric grid (20 million plus values)



Measurement-Based Modal Analysis

- There are a number of different approaches
- The idea of all techniques is to approximate a signal, $y_{org}(t)$, by the sum of other, simpler signals (basis functions)
  - Basis functions are usually exponentials, with linear and quadratic functions used to detrend the signal
  - Properties of the original signal can be quantified from basis function properties
    - Examples are frequency and damping
  - Signal is considered over time with t=0 as the start
- Approaches sample the original signal $y_{org}(t)$

Measurement-Based Modal Analysis 2

- Vector **y** consists of m uniformly sampled points from $y_{org}(t)$ at a sampling value of ΔT, starting with t=0, with values y_j for j=1…m
  - Times are then $t_j = (j-1)\Delta T$
  - At each time point j, the approximation of y_j is

$$\hat{y}_j(\mathrm{t}_j, \boldsymbol{\alpha}) = \sum_{i=1}^{n} b_i \phi_i(t_j, \boldsymbol{\alpha})$$

where $\boldsymbol{\alpha}$ is a vector with the real and imaginary eigenvalue components, with $\phi_i(t_j, \boldsymbol{\alpha}) = e^{\alpha_i t_j}$ for α_i corresponding to a real eigenvalue, and $\phi_i(t_j, \boldsymbol{\alpha}) = e^{\alpha_i t_j} \cos(\alpha_{\mathrm{i}+1} \mathrm{t}_j)$ and $\phi_{i+1}(\boldsymbol{\alpha}) = e^{\alpha_i t_j} \sin(\alpha_{\mathrm{i}+1} \mathrm{t}_j)$ for a complex eigenvector value

Measurement-Based Modal Analysis 3

- Error (residual) value at each point j is

$$r_j(t_j, \boldsymbol{\alpha}) = y_j - \hat{y}_j(t_j, \boldsymbol{\alpha})$$

- The closeness of the fit can be quantified using the Euclidean norm of the residuals

$$\frac{1}{2}\sum_{j=1}^{m}(y_j - \hat{y}_j(t_j, \boldsymbol{\alpha}))^2 = \frac{1}{2}\|\mathbf{r}(\boldsymbol{\alpha})\|_2^2$$

- Hence we need to determine α and **b**

$$\hat{y}_j(\mathrm{t}_j, \boldsymbol{\alpha}) = \sum_{i=1}^{n} b_i \phi_i(t_j, \boldsymbol{\alpha})$$

Sampling Rate and Aliasing

- The Nyquist-Shannon sampling theory requires sampling at twice the highest desired frequency
  - For example, to see a 5 Hz frequency we need to sample the signal at a rate of at least 10 Hz
- Sampling shifts the frequency spectrum by 1/T (where T is the sample time), which causes frequency overlap
- This is known as aliasing, which can cause a high frequency signal to appear to be a lower frequency signal
  - Aliasing can be reduced by fast sampling and/or low pass filters

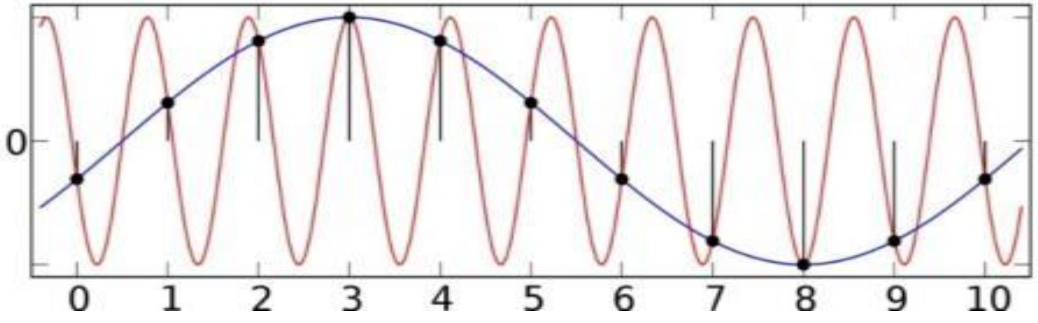


Image: upload.wikimedia.org/wikipedia/commons/thumb/2/28/AliasingSines.svg/2000px-AliasingSines.svg.png

One Solution Approach: The Matrix Pencil Method

- There are several algorithms for finding the modes. We'll use the Matrix Pencil Method
  - This is a newer technique for determining modes from noisy signals (from about 1990, introduced to power system problems in 2005); it is an alternative to the Prony Method
  - The Matrix Pencil Method is useful when there is signal noise
- Given m samples, with L=m/2, the first step is to form the Hankel Matrix, **Y** such that

This not a sparse matrix

$$\mathbf{Y} = \begin{bmatrix} y_1 & y_2 & \cdots & y_{L+1} \\ y_2 & y_3 & \cdots & y_{L+2} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m-L} & y_{m-L+1} & \cdots & y_m \end{bmatrix}$$

Reference: A. Singh and M. Crow, "The Matrix Pencil for Power System Modal Extraction," IEEE Transactions on Power Systems, vol. 20, no. 1, pp. 501-502, Feb 2005.

Algorithm Details, cont.

- Then calculate **Y**'s singular values using an economy singular value decomposition (SVD)

$$\mathbf{Y} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathbf{T}}$$

- The ratio of each singular value is then compared to the largest singular value σ_c; retain the ones with a ratio > than a threshold
  - This determines the modal order, M
  - Assuming **V** is ordered by singular values (highest to lowest), let $\mathbf{V}_p$ be then matrix with the first M columns of **V**

The computational complexity increases with the cube of the number of measurements!

This threshold is a value that can be changed; decrease it to get more modes.

Aside: The Matrix Singular Value Decomposition (SVD)

- The SVD is a factorization of a matrix that generalizes the eigendecomposition to any m by n matrix to produce

$$\mathbf{Y} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathbf{T}}$$

The original concept is more than 100 years old, but has found lots of recent applications

- where Σ is a diagonal matrix of the singular values
- The singular values are non-negative, real numbers that can be used to indicate the major components of a matrix (the gist is they provide a way to decrease the rank of a matrix)

Aside: SVD Image Compression Example

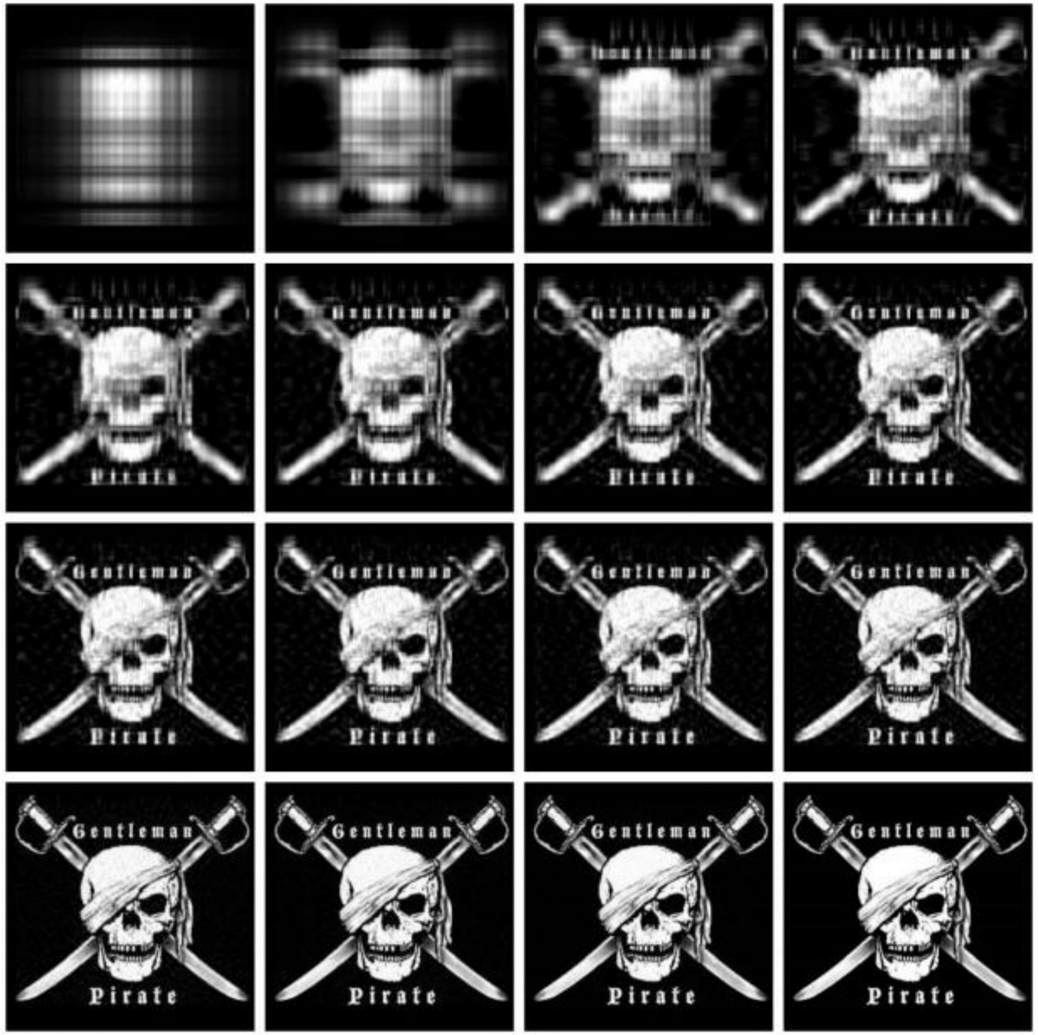


Figure 3.1: Image size 250x236 – modes used {{1,2,4,6},{8,10,12,14},{16,18,20,25},{50,75,100,original image}}

Images can be represented with matrices. When an SVD is applied and only the largest singular values are retained the image is compressed.

Image Source: www.math.utah.edu/~goller/F15_M2270/BradyMathews_SVDImage.pdf

Matrix Pencil Algorithm Details, cont.

- Then form the matrices $\mathbf{V}_1$ and $\mathbf{V}_2$ such that
  - $\mathbf{V}_1$ is the matrix consisting of all but the last row of $\mathbf{V}_p$
  - $\mathbf{V}_2$ is the matrix consisting of all but the first row of $\mathbf{V}_p$
- Discrete-time poles are found as the generalized eigenvalues of the pair $(\mathbf{V}_2^T\mathbf{V}_1, \mathbf{V}_1^T\mathbf{V}_1) = (\mathbf{A},\mathbf{B})$
- These eigenvalues are the discrete-time poles, z_i with the modal eigenvalues then

$$\lambda_i = \frac{\ln(z_i)}{\Delta T}$$

If **B** is nonsingular (the situation here) then the generalized eigenvalues are the eigenvalues of $\mathbf{B}^{-1}\mathbf{A}$

The log of a complex number $z=r\angle\theta$ is $\ln(r) + j\theta$

Matrix Pencil Method with Many Signals

- The Matrix Pencil approach can be used with one signal or with multiple signals
- Multiple signals are handled by forming a $\mathbf{Y}_k$ matrix for each signal k using the measurements for that signal and then combining the matrices

$$\mathbf{Y}_k = \begin{bmatrix} y_{1,\mathrm{k}} & y_{2,\mathrm{k}} & \cdots & y_{L+1,\mathrm{k}} \\ y_{2,\mathrm{k}} & y_{3,k} & \cdots & y_{L+2,\mathrm{k}} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m-L,\mathrm{k}} & y_{m-L+1,\mathrm{k}} & \cdots & y_{m,\mathrm{k}} \end{bmatrix}$$

$$\mathbf{Y} = \begin{bmatrix} \mathbf{Y}_1 \\ \vdots \\ \mathbf{Y}_N \end{bmatrix}$$

The required computation scales linearly with the number of signals

Matrix Pencil Method with Many Signals 2

- However, when dealing with many signals, usually the signals are somewhat correlated, so vary few of the signals are actually need to be included to determine the desired modes
- Ultimately we are finding

$$y_j(\mathrm{t}_j, \boldsymbol{\alpha}) = \sum_{i=1}^{n} b_i \phi_i(t_j, \boldsymbol{\alpha})$$

- The α is common to all the signals (i.e., the system modes) while the **b** vector is signal specific (i.e., how the modes manifest in that signal)

Quickly Determining the b Vectors

- A key insight is from an approach known as the Variable Projection Method (from Borden, 2013) that for any signal k

  $\mathbf{y}_k = \boldsymbol{\Phi}(\boldsymbol{\alpha})\mathbf{b}_k$

  And then the residual is minimized by selecting $\mathbf{b}_k = \boldsymbol{\Phi}(\boldsymbol{\alpha})^{+}\mathbf{y}_k$

  where $\boldsymbol{\Phi}(\boldsymbol{\alpha})$ is the m by n matrix with values

  $\Phi_{ji}(\boldsymbol{\alpha}) = e^{\alpha_i t_j}$ if α_i corresponds to a real eigenvalue,

  and $\Phi_{ji}(\boldsymbol{\alpha}) = e^{\alpha_i t_j}\cos(\alpha_{i+1} t_j)$ and $\Phi_{ji+1}(\boldsymbol{\alpha}) = e^{\alpha_i t_j}\sin(\alpha_{i+1} t_j)$

  for a complex eigenvalue; $t_j = (j-1)\Delta T$

  Finally, $\boldsymbol{\Phi}(\boldsymbol{\alpha})^{+}$ is the pseudoinverse of $\boldsymbol{\Phi}(\boldsymbol{\alpha})$

Where m is the number of measurements and n is the number of modes

A. Borden, B.C. Lesieutre, J. Gronquist, "Power System Modal Analysis Tool Developed for Industry Use," *Proc. 2013 North American Power Symposium*, Manhattan, KS, Sept. 2013

Aside: Pseudoinverse of a Matrix

- The pseudoinverse of a matrix generalizes concept of a matrix inverse to an m by n matrix, in which m >= n
  - Specifically this is a Moore-Penrose Matrix Inverse
- Notation for the pseudoinverse of **A** is $\mathbf{A}^{+}$
- Satisfies $\mathbf{A}\mathbf{A}^{+}\mathbf{A} = \mathbf{A}$
- If **A** is a square matrix, then $\mathbf{A}^{+} = \mathbf{A}^{-1}$
- Quite useful for solving the least squares problem since the least squares solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = \mathbf{A}^{+}\mathbf{b}$
- Can be calculated using an SVD

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{T}$$
$$\mathbf{A}^{+} = \mathbf{V}\boldsymbol{\Sigma}^{+}\mathbf{U}^{T}$$

Least Squares Matrix Pseudoinverse Example

- Assume we wish to fix a line (mx + b = y) to three data points: (1,1), (2,4), (6,4)
- Two unknowns, m and b; hence **x** = [m b]^T
- Setup in form of **Ax** = **b**

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix} \quad \text{so } \mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 6 & 1 \end{bmatrix}$$

Least Squares Matrix Pseudoinverse Example, cont.

- Doing an economy SVD

$$\mathbf{A} = \mathbf{U\Sigma V}^T = \begin{bmatrix} -0.182 & -0.765 \\ -0.331 & -0.543 \\ -0.926 & 0.345 \end{bmatrix} \begin{bmatrix} 6.559 & 0 \\ 0 & 0.988 \end{bmatrix} \begin{bmatrix} -0.976 & -0.219 \\ 0.219 & -0.976 \end{bmatrix}$$

- Computing the pseudoinverse

$$\mathbf{A}^+ = \mathbf{V}\,\mathbf{\Sigma}^+\mathbf{U}^T = \begin{bmatrix} -0.976 & 0.219 \\ -0.219 & -0.976 \end{bmatrix} \begin{bmatrix} 0.152 & 0 \\ 0 & 1.012 \end{bmatrix} \begin{bmatrix} -0.182 & -0.331 & -0.926 \\ -0.765 & -0.543 & 0.345 \end{bmatrix}$$

$$\mathbf{A}^+ = \mathbf{V}\,\mathbf{\Sigma}^+\mathbf{U}^T = \begin{bmatrix} -0.143 & -0.071 & 0.214 \\ 0.762 & 0.548 & -0.310 \end{bmatrix}$$

In an economy SVD the **S** matrix has dimensions of m by m if m < n or n by n if n < m

Least Squares Matrix Pseudoinverse Example, cont. 2

- Computing **x** = [m b]^T gives

$$\mathbf{A}^{+}\mathbf{b} = \begin{bmatrix} -0.143 & -0.071 & 0.214 \\ 0.762 & 0.548 & -0.310 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 0.429 \\ 1.71 \end{bmatrix}$$

- With the pseudoinverse approach we immediately see the sensitivity of the elements of **x** to the elements of **b**
  - New values of m and b can be readily calculated if **y** changes
- Computationally the SVD is order mn^2+n^3 (with n < m)
  - In this example it means it scales linearly with the number of points; matrices with m >> n are common

Computational Considerations

- When there is just one signal, the procedure scales with the cube of the number of measurements
  - This value is usually relatively small, say 20 seconds of data sampled at 10 Hz for 200 measurements
- If multiple signals are included, it scales linearly with the number of signals
- However, a key insight is once α has been determined, each $\mathbf{b}_k$ can be determined with a matrix multiply of a matrix with dimensions of the number of modes and number of measurements

$$\mathbf{y}_k = \boldsymbol{\Phi}(\boldsymbol{\alpha})\mathbf{b}_k \rightarrow \mathbf{b}_k = \boldsymbol{\Phi}(\boldsymbol{\alpha})^{+}\mathbf{y}_k$$

$$\boldsymbol{\Phi}(\boldsymbol{\alpha})^{+} \text{ is the pseudoinverse of } \boldsymbol{\Phi}(\boldsymbol{\alpha})$$

We can quickly determine how well **a** matches each signal