ECEN 667
Power System Stability

Lecture 3: Numerical Integration of Differential Equations

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Announcements

- Homework Assignment #1 is due Thursday, Sept. 11th at 8 AM. Email me your solution as a single PDF.
- Review the slides and PowerWorld examples

Some Definitions

- **Ordinary Differential Equations (ODEs) (Explicit, First-Order)**

  A set of equations with only total derivatives

$$\dot{x} = f(x)$$

- **Differential Algebraic Equations (DAEs)**

  A set of equations that includes some non-differential equations

$$\dot{x} = f(x, y)$$
$$0 = g(x, y)$$

- **Partial Differential Equations (PDEs)**

  Includes partial derivatives, not used in power system stability

$$\frac{\partial y}{\partial t} = f(y, x, t) \qquad \frac{\partial y}{\partial x} = f(y, x, t)$$

- **Initial Value Problem (IVP)** is a set of ODEs or DAEs with an initial values $x(0) = x_0 \quad y(0) = y_0$ from which the goal is to find future values of x and y

What's the Solution to this Differential Equation?

$$\dot{x}(t) = f(t) = -x(t),\ x(0) = 1$$

- Analytical solution: $x(t) = e^{-t}$
- Numerical methods: solving successive time values separated by Δt
  - **Euler**, or Forward Euler, is the simplest explicit method
  - **Backward Euler**, is the simplest implicit method
  - **Trapezoidal**, implicit method sort of like an average of Forward and Backward Euler
  - **Runge-Kutta 2**, explicit method based on further expansion of Taylor series
  - **Runge-Kutta 4**, extended version of RK2
  - **Adams-Bashforth**, a multi-step explicit method
  - Other methods…
- Euler's method says: $x(t + \Delta t) = x(t) + \Delta t \cdot f(t)$
  - So for this example, $x(t + \Delta t) = x(t) + \Delta t \cdot \big(-x(t)\big) = x(t) \cdot (1 - \Delta t)$
  - Table on right shows for $\Delta t = 0.1$

t	Analytical	Euler
0	1	1
0.1	0.90484	0.90000
0.2	0.81873	0.81000
0.3	0.74082	0.72900
0.4	0.67032	0.65610
0.5	0.60653	0.59049
0.6	0.54881	0.53144

Definition of Other Methods

- **Backward Euler**, is the simplest implicit method
  - Definition: $x(t+\Delta t) = x(t) + \Delta t \cdot f(t + \Delta t)$
  - This means we have to do some algebraic manipulation (implicit method)
  - For our example: $x(t+\Delta t) = x(t) + \Delta t \cdot \left(-x(t+\Delta t)\right)$
  - Solve for $x(t+\Delta t) = x(t) \cdot \left(\frac{1}{1+\Delta t}\right)$
- **Trapezoidal**, implicit method averaging Forward and Backward Euler
  - Definition: $x(t+\Delta t) = x(t) + \frac{\Delta t}{2}\left(f(t) + f(t+\Delta t)\right)$
  - For our example: $x(t+\Delta t) = x(t) + \frac{\Delta t}{2} \cdot \left(-x(t) - x(t+\Delta t)\right)$
  - Solve again: $x(t+\Delta t) = x(t)\frac{1-\Delta t/2}{1+\Delta t/2}$
- For both of these methods, we rely on being able to solve back for $x(t+\Delta t)$, which may not be easy for complicated non-linear functions
- Theoretical error for any of these first three is $O((\Delta t)^2)$

Definition of Other Methods, Continued

- **Runge-Kutta 2**, explicit method based on second-order approximation of Taylor series (considering f as a function of both x and t)

  $k_1 = \Delta t \cdot f(x(t), t)$ (This is the same as Euler term)

  $k_2 = \Delta t \cdot f(x(t) + k_1, t + \Delta t)$

  $x(t + \Delta t) = x(t) + \frac{1}{2}k_1 + \frac{1}{2}k_2$

  Derivation of this is given on the next two slides, for reference.

- **Runge-Kutta 4**, extended version of RK2 to the fourth-order polynomial

  $k_1 = \Delta t \cdot f(x(t), t)$

  $k_2 = \Delta t \cdot f(x(t) + k_1/2, t + \Delta t/2)$

  $k_3 = \Delta t \cdot f(x(t) + k_2/2, t + \Delta t/2)$

  $k_4 = \Delta t \cdot f(x(t) + k_3, t + \Delta t)$

  $x(t + \Delta t) = x(t) + (k_1 + 2k_2 + 2k_3 + k_4)/6$

Derivation of the Runge-Kutta 2nd Order Model

- Take the Taylor series of $x(t + \Delta t)$ out to the second order

$$x(t+\Delta t) = x(t) + \Delta t \cdot f(x(t),t) + \left(\frac{\Delta t^2}{2}\right)\frac{df}{dt}(x(t),t) + \mathrm{O}((\Delta t)^3)$$

Remember $f = \frac{dy}{dt}$

$$= x(t) + \Delta t \cdot f(x(t),t) + \left(\frac{\Delta t^2}{2}\right)\left[\frac{\partial f}{\partial x}f + \frac{\partial f}{\partial t}\right](x(t),t) + \mathrm{O}((\Delta t)^3)$$

- Now try to find a method with the following form

$$k_1 = \Delta t \cdot f(x(t),t)$$

$$k_2 = \Delta t \cdot f(x(t) + \beta k_1, t + \alpha \Delta t)$$

$$x(t+\Delta t) = x(t) + a k_1 + b k_2$$

- Expand k_2 with a multi-variable Taylor series

$$k_2 = \Delta t\left(f(x(t),t) + \frac{\partial f}{\partial x}\beta k_1 + \frac{\partial f}{\partial x}\alpha \Delta t + O\big((\Delta t)^2\big)\right)$$

Derivation of the RK2, continued

- Now plug in expanded version of k2 and k1

$$x(t+\Delta t) = x(t) + a\Delta t \cdot f(x(t),t) + b\Delta t \left(f(x(t),t) + \frac{\partial f}{\partial x}\beta k_1 + \frac{\partial f}{\partial x}\alpha\Delta t + O\big((\Delta t)^2\big) \right)$$

$$= x(t) + (a+b)\Delta t \cdot f(x(t),t) + \beta b(\Delta t)^2 \frac{\partial f}{\partial x} f + \alpha b(\Delta t)^2 \frac{\partial f}{\partial x} + O\big((\Delta t)^3\big)$$

- Comparing coefficients to the full Taylor series, we get $a+b=1; \beta b = \frac{1}{2}; \alpha b = \frac{1}{2}$
- Arbitrarily choose $b = \frac{1}{2}$ and we get $a = \frac{1}{2}; \alpha = 1; \beta = 1$
- Resulting formulation has error of $O\big((\Delta t)^3\big)$
- Similar derivation for fourth-order Runge-Kutta, with error $O\left((\Delta t)^5\right)$
- These are explicit methods, no need to solve algebraic equations

Second Order Runge-Kutta Example

- Consider the "oscillating cart" problem

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1$$

- With an initial value of $x_1(0) = 1; x_2(0) = 0$ there is an analytical solution of $x_1(t) = \cos(t)\,; x_2(t) = -\sin(t)$
- But let's do it numerically with RK2 and $\Delta t = 0.25$

$$\mathrm{k}_1 = \Delta t\, f(x) = (0.25)\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.25 \end{bmatrix}$$

$$x(0) + \mathrm{k}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -0.25 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.25 \end{bmatrix}$$

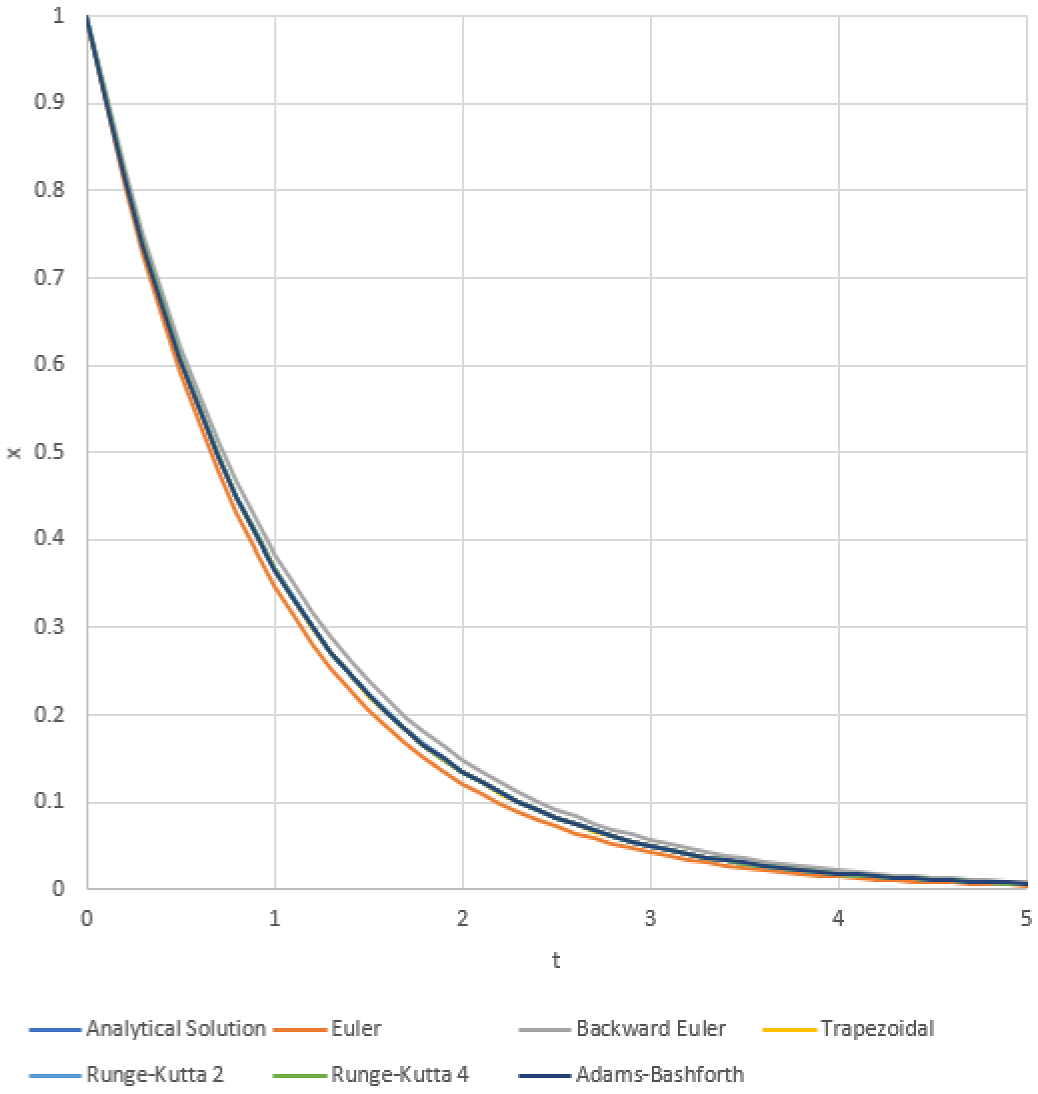
$$\mathrm{k}_2 = \Delta t\, f(x(0) + \mathrm{k}_1) = \begin{bmatrix} -0.0625 \\ -0.25 \end{bmatrix}$$

$$x(0.25) = x(0) + \frac{1}{2}(\mathrm{k}_1 + \mathrm{k}_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{2}\left(\begin{bmatrix} 0 \\ -0.25 \end{bmatrix} + \begin{bmatrix} -0.0625 \\ -0.25 \end{bmatrix}\right) = \begin{bmatrix} 0.96875 \\ -0.25 \end{bmatrix}$$

Definition of Other Methods, Continued

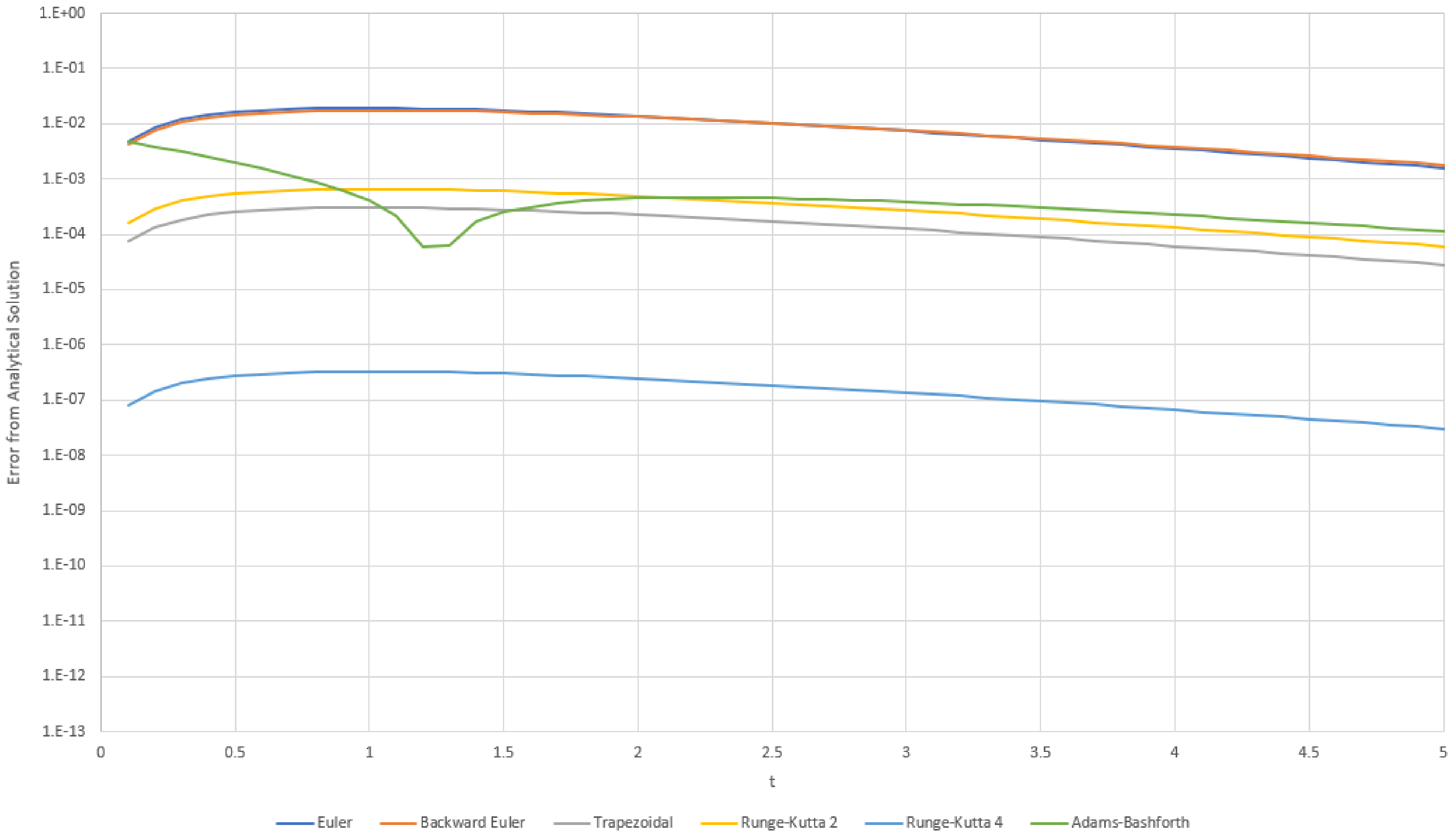
- **Adams-Bashforth**, a multi-step explicit method
  - Definition: $x(t+\Delta t) = x(t) + \frac{\Delta t}{2}\left(3f(x(t),t) - f(x(t-\Delta t), t-\Delta t)\right)$
  - Notice it uses the prior time step $(t-\Delta t)$
  - This is good computationally since we should have this data already.
  - But one issue is nonlinearities and discontinuities
  - Error is order $O\left((\Delta t)^3\right)$
- There are many other methods, including Adams-Moulton, predictor-corrector, along with higher-order RK methods

Numerical Experiments



t	Analytical Solution	Euler	Backward Euler	Trapezoidal	Runge-Kutta 2	Runge-Kutta 4	Adams-Bashforth
0	1	1	1	1	1	1	1
0.1	0.904837	0.9	0.909091	0.904762	0.905	0.904838	0.9
0.2	0.818731	0.81	0.826446	0.818594	0.819025	0.818731	0.815
0.3	0.740818	0.729	0.751315	0.740633	0.741218	0.740818	0.73775
0.4	0.67032	0.6561	0.683013	0.670096	0.670802	0.67032	0.667838
0.5	0.606531	0.59049	0.620921	0.606278	0.607076	0.606531	0.604549
0.6	0.548812	0.531441	0.564474	0.548537	0.549404	0.548812	0.547259
0.7	0.496585	0.478297	0.513158	0.496295	0.49721	0.496586	0.495397
0.8	0.449329	0.430467	0.466507	0.449029	0.449975	0.449329	0.448451
0.9	0.40657	0.38742	0.424098	0.406264	0.407228	0.40657	0.405953
1	0.367879	0.348678	0.385543	0.367573	0.368541	0.36788	0.367483
1.1	0.332871	0.313811	0.350494	0.332566	0.33353	0.332871	0.332658
1.2	0.301194	0.28243	0.318631	0.300893	0.301844	0.301195	0.301133
1.3	0.272532	0.254187	0.289664	0.272236	0.273169	0.272532	0.272596
1.4	0.246597	0.228768	0.263331	0.246309	0.247218	0.246597	0.246763
1.5	0.22313	0.205891	0.239392	0.222851	0.223732	0.22313	0.223379
1.6	0.201897	0.185302	0.217629	0.201627	0.202478	0.201897	0.20221
1.7	0.182684	0.166772	0.197845	0.182425	0.183242	0.182684	0.183048
1.8	0.165299	0.150095	0.179859	0.165051	0.165834	0.165299	0.165701
1.9	0.149569	0.135085	0.163508	0.149332	0.15008	0.149569	0.149998
2	0.135335	0.121577	0.148644	0.13511	0.135822	0.135336	0.135783

Error Analysis



Summary of Methods

Name	Type	Error	Notes
Euler	Explicit	$O((\Delta t)^2)$	
Backward Euler	Implicit	$O((\Delta t)^2)$	Because it is implicit, requires backsolving algebraic equations
Trapezoidal	Implicit	$O((\Delta t)^2)$	Because it is implicit, requires backsolving algebraic equations. Often better than backward Euler
Runge-Kutta 2	Explicit	$O((\Delta t)^3)$	Requires differential equations be evaluated twice per time step
Runge-Kutta 4	Explicit	$O((\Delta t)^5)$	Requires differential equations be evaluated four times per time step
Adams-Bashforth	Explicit	$O((\Delta t)^3)$	Uses prior time step solution, which may cause errors with discontinuities

An advantage of implicit methods is they are numerically stable. But that doesn't mean explicit methods wouldn't be appropriate in many applications.

Note there are many versions of Runge-Kutta methods, some of which are implicit

Equilibrium Point

- A differential equation's equilibrium point x^* is a point at which the variables are not changing
  - To find them, solve for the conditions in which the derivatives are zero
$$\dot{x} = f(x^*) = 0$$
  - Equation sets may have multiple equilibrium points
- An equilibrium point is Lyapunov stable if the response to a small disturbance remains small
  - That is, $\text{for all } \epsilon > 0, \exists\delta > 0$ such that if $x(0) - x^* < \delta$ then $x(t) - x^* < \epsilon$ for $t \geq 0$
- An equilibrium point has asymptotic stability if there exists a $\delta > 0 \text{ such}$ that if $x(0) - x^* < \delta$ then $\lim_{t\to\infty} |x(t) - x^*| = 0$

Stiffness in Differential Equations

- A set of ODEs or DAEs is considered "stiff" if some phenomena of interest are occurring rather quickly with respect to some other phenomena
- This affects the solution efficiency; we often need to take quite small time steps to account for the fast dynamics
- Example

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -1000x_1 - 1001x_2$$

General solution is

$$\dot{x}_1(t) = Ae^{-t} + Be^{-1000t}$$

What's the Best Numerical Method?

- It depends on the application
- Some issues to consider
  - Speed
  - Accuracy
  - Numerical stability
  - Code complexity (such as implicit/explicit)
  - Discontinuities, which can be difficult for multi-step methods
- Historically, EMT simulations have generally used the trapezoidal method
  - Numerical stability is important with stiff equations and sinusoidal steady-state conditions
- Historically, stability problems (as in this class), have generally used explicit methods such as forward Euler, Second Order Runge-Kutta, and Adams-Bashforth

Returning to Example 1

$$\dot{\delta} = \omega \cdot \omega_s$$

$$\dot{\omega} = \frac{1}{6}\left(\frac{1}{\omega+1} - \frac{1.0838}{0.1}(V_r \sin\delta - V_i \cos\delta)\right)$$

$$1.0838\angle\delta - \bar{V} = j0.1\bar{I}$$

$$\bar{V} - 1.05\angle 0 = (0.05 + j0.2)\bar{I}$$

- Note how this maps onto the generic formulation

$$\dot{x} = f(x, y)$$
$$0 = g(x, y)$$

- The first two equations (generator dynamics) are the differential equations (f), and the last two equations (network solution) are the algebraic equations (g).

DAE Numerical Solutions

- A typical method for solving DAEs with an explicit method such as Runge-Kutta involves alternating between explicit and implicit algebraic calculations
  - Given a value of x, we want to find $\dot{x}$ for RK2 or another method
  - So first we do an implicit algebraic solution of $0 = g(x, y)$ given our value of x and find the corresponding value of y. (In power systems this is essentially solving the network equations.)
  - Then, once we have both x and y, we do an explicit evaluation of $f(x, y)$.
- Next time, let's try it for Example 1